Transformation of general curve evolution to a modified Belavin–Polyakov equation

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We show that the general evolution of the tangent to a curve in three-dimensional space can be transformed to a modified form of the Belavin–Polyakov equation. Using this, we find a rich variety of exact instanton and twist solutions for several classes of evolution. Certain physical applications are also discussed. © *1997 American Institute of Physics.* [S0022-2488(97)02410-9]

I. INTRODUCTION

Many problems in physics can be modeled in terms of curves in three-dimensional space. A vortex filament in a fluid,[1] a particle trajectory, and a polymer chain are obvious examples of space curves. Less obvious an example is the magnetic moment vector along a classical magnetic spin chain, where the magnetic moment can be regarded as defining the tangent to some space curve.[2] The study of the evolution of a space curve is therefore useful in many physical applications. Several years ago, Lamb[3] analyzed the equations for a moving curve represented by two sets of Frenet–Serret equations[4] for the tangent, normal, and binormal vectors to the curve. On imposing compatibility conditions on these vectors, coupled nonlinear partial differential equations for the curvature and torsion of the curve can be obtained. He showed that under certain conditions these turn out to be integrable, soliton-bearing equations[5] such as the nonlinear Schrödinger equation, sine-Gordon equation, etc., indicating that the underlying curve evolution is also integrable. In such cases, a method proposed by Sym[6] can be used in principle to obtain the solution to the curve evolution, using the Lax pair of the corresponding soliton equation. In general, though, the reconstruction of a three-dimensional evolving curve using the solutions for the corresponding curvature and the torsion is a nontrivial task. In recent years, there has been renewed interest in such geometric connections and their various ramifications.[7]

In this paper, we adopt a different approach to this problem. Instead of analyzing the solvability of the coupled (scalar) equations for the curvature and torsion, we ask under what conditions the fundamental (vector) equations of curve evolution (viz., the two sets of Frenet–Serret equations) can themselves be reduced to a solvable form. The advantage of this approach is that the moving three-dimensional curve can be constructed *directly* from the known solution of the evolving tangent vector. We first show that for a wide class of evolutions, a *modified* form of the Belavin–Polykov equation[8] for the tangent arises in a natural fashion. Using this result, we proceed to analyze new special classes of evolution kinematics. In particular, we find that in transformed coordinates the solution for the tangent vector takes on the form of instanton and twist solutions. Physical applications to the kinematics of a polymer chain and the dynamics of an inhomogeneous antiferromagnetic chain are discussed.

II. GENERAL CURVE EVOLUTION EQUATIONS

Let us consider a curve embedded in three-dimensional space, described in parametric form by a position vector $\mathbf{r}=\mathbf{r}(s)$, s being the usual arclength variable.[4] Let $\mathbf{t}=\mathbf{r}_s$ be the unit tangent

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vector along the curve. We denote by **n** and **b**, respectively, the principal normal and binormal to the curve. The triad of unit vectors (**t**,**n**,**b**) represents a locally orthonormal system that is known to satisfy the Frenet–Serret equations,[4]

$$\mathbf{t}_s = K\mathbf{n}, \quad \mathbf{n}_s = -K\mathbf{t} + \tau\mathbf{b}, \quad \mathbf{b}_s = -\tau\mathbf{n}, \tag{2.1}$$

where the subscripts denote d/ds. The curvature K and the torsion τ are given by

$$K = (\mathbf{t}_s \cdot \mathbf{t}_s)^{1/2}, \quad \tau = \mathbf{t} \cdot (\mathbf{t}_s \times \mathbf{t}_{ss})/K^2. \tag{2.2}$$

We now consider the evolution of this curve with time u, so that $\mathbf{r} = \mathbf{r}(s,u)$. The evolution of the triad (**t**,**n**,**b**) can be written quite generally in a form similar to the Frenet–Serret set (2.1):

$$\mathbf{t}_u = g\mathbf{n} + h\mathbf{b}, \quad \mathbf{n}_u = -g\mathbf{t} + \tau_0\mathbf{b}, \quad \mathbf{b}_u = -h\mathbf{n} - \tau_0\mathbf{n}. \tag{2.3}$$

The scalars g, h, and τ_0 along with appropriate boundary conditions completely determine the dynamics of the curve. Note that in the time evolution of **t** there is an additional term in the **b** direction. The reason for the absence of such a term in the space derivatives is that one has the freedom to align $\mathbf{t}_s$ in the direction of the normal **n**, but once this is done, the time derivative can, in general, have both **n** and **b** components. In the following discussion we shall limit ourselves to nonstretching curves, requiring that the unit triad satisfy the compatibility conditions

$$\mathbf{t}_{us} = \mathbf{t}_{su}, \quad \mathbf{n}_{us} = \mathbf{n}_{su}, \quad \text{and} \quad \mathbf{b}_{us} = \mathbf{b}_{su}.$$

With straightforward manipulations, these conditions can be shown to lead to the following relations between the above scalars:

$$K_u = g_s - \tau h, \quad \tau_u = (\tau_0)_s + Kh, \quad h_s = (K\tau_0 - \tau g). \tag{2.4}$$

The three Eqs. (2.4) relate the five ''curvatures'' K, τ, g, h, and τ_0, suggesting that only two of these scalar functions are independent. Indeed, we shall see below that, in order to specify the evolution of the tangent vector **t**, all we need is two combinations of the quantities g, K, and h. We now note that Eqs. (2.1)–(2.3) imply quite generally, the following vector relation:

$$\mathbf{t}_s \times \mathbf{t}_u = Kh\mathbf{t}. \tag{2.5}$$

Taking the cross product of (2.5) from the left with $\mathbf{t}_s$, and recalling that we have $\mathbf{t} \cdot \mathbf{t}_s = \mathbf{t} \cdot \mathbf{t}_u = 0$ (because $|\mathbf{t}|^2 = 1$), we obtain

$$(\mathbf{t}_s \cdot \mathbf{t}_u)\mathbf{t}_s - (\mathbf{t}_s \cdot \mathbf{t}_s)\mathbf{t}_u = Kh(\mathbf{t}_s \times \mathbf{t}). \tag{2.6}$$

From Eqs. (2.1) and (2.3), we identify

$$\mathbf{t}_s \cdot \mathbf{t}_u = Kg, \quad \mathbf{t}_s \cdot \mathbf{t}_s = K^2, \quad \mathbf{t}_u \cdot \mathbf{t}_u = g^2 + h^2. \tag{2.7}$$

Substituting relations (2.7) in (2.6), and excluding the trivial case $K=0$, we have

$$g\mathbf{t}_s - K\mathbf{t}_u = h(\mathbf{t}_s \times \mathbf{t}). \tag{2.8}$$

Taking the cross product of Eq. (2.8) with **t** twice in succession yields the two equations

$$\begin{aligned} \mathbf{t}_s &= (\beta\mathbf{t}_u - \alpha\mathbf{t}_s) \times \mathbf{t}, \\ \mathbf{t}_u &= (\alpha\mathbf{t}_u - [(\alpha^2+1)/\beta]\mathbf{t}_s) \times \mathbf{t}, \end{aligned} \tag{2.9}$$

where $\alpha = g/h$, $\beta = K/h$. Here h is assumed to be nonzero. (The special case $h=0$ is treated separately below.) Although the second equation in (2.9) is obtained directly from the first and is therefore not independent, we write it down explicitly to simplify later manipulations. Thus, any one of the relations (2.8) and (2.9) describes the general evolution of an arbitrary curve in three dimensions and is in effect the starting point of our analysis.

Before getting down to the analysis of the general case, we first discuss two special cases for which exact solutions can be obtained. These cases are not only of interest in their own right, but are also of help in understanding the more general treatment to follow.

III. SPECIAL CASES AND THE MODIFIED BELAVIN–POLYAKOV EQUATION (MBPE)

Case (i): Suppose for all u, the space–curve evolution is such that

$$h=0; \quad K,g \neq 0.$$

In this case, Eq. (2.8) yields

$$\mathbf{t}_u = (g/K)\mathbf{t}_s. \tag{3.1}$$

We now observe that if the scalar function g/K is separable, namely,

$$g/K = G(u)/F(s) \tag{3.2}$$

where F and G are arbitrary integrable functions, then Eq. (3.1) becomes a *linear* equation for $\mathbf{t}$:

$$\mathbf{t}_{u'} = \mathbf{t}_{s'}, \tag{3.3}$$

where the functions $s'(s)$ and $u'(u)$ are defined by $s' = \int^s F(s)ds$ and $u' = \int^u G(u)du$. It is evident that, in this case, the components of $\mathbf{t}$ can be any arbitrary functions of the variable $\zeta = s' + u'$. Differently interpreted, any initial form of $\mathbf{t}(s'(s), u'(u=0))$ on the curve with the parametrization s' moves with (transformed) time along the curve without changing its shape with a dimensionless velocity equal to -1. Now, since $\mathbf{t}$ is a unit vector, it can be written in spherical polar coordinates as

$$\mathbf{t} = (\sin\,\theta\,\cos\,\phi,\;\; \sin\,\theta\,\sin\,\phi,\;\; \cos\,\theta), \tag{3.4}$$

where θ and ϕ are the polar and azimuthal angles. For illustration, let us examine the nontrivial case $F(s) = s^x$ and $G(u) = u^y$, say, where x,y are odd integers >1. Suppose further that at $u=0$, $\mathbf{t}$ had the (arbitrarily chosen) form

$$\cos\,\theta \;=\; \exp(-s'^2/2\sigma^2), \quad \phi = \cos\,s',$$

where σ is a constant determining the width of the Gaussian. Here, the curve is a straight line at $s \to \pm\infty (\theta = \pi/2)$, and as one approaches $s=0$ the tangent vector tends asymptotically to the upward direction $(\theta \to 0)$. The curve is also turning periodically in s', which means that in the real space coordinates the pitch decreases as a power, $x+1$, of the distance from $s=0$. At any later time the angles will develop according to

$$\begin{aligned} \cos\,\phi \;&=\; \exp\{-[s^{x+1}/(x+1) + u^{y+1}/(y+1)]^2/2\sigma^2\}, \\ \phi \;&=\; \cos[s^{x+1}/(x+1) + u^{y+1}/(y+1)]. \end{aligned} \tag{3.5}$$

Thus, the initial maximum in $\cos\,\theta$ moves along the curve at a velocity that is position-dependent $(\sim s^{1-x/y})$. In addition to this longitudinal sliding, the tangent continuously turns along the curve in the perpendicular plane, viz., at any given location along the curve the tangent rotates as a function of time with the rotation frequency increasing as a power, $y+1$, of time. This exotic solution is but one example of a family of solutions that can be realized in this case.

Case (ii): Now suppose that for all u the curve evolution is such that

$$g=0; \quad K,h\neq 0.$$

Equations (2.9) become

$$\mathbf{t}_s=(1/f\ )(\mathbf{t}_u\times\mathbf{t}), \quad \mathbf{t}_u=f(\mathbf{t}\times\mathbf{t}_s), \tag{3.6}$$

where $f=(1/\beta)=(h/K)$. We call equations of the forms in (3.6) the modified Belavin–Polyakov equation (MBPE), owing to the resemblance to the known Belavin–Polyakov equation (BPE), as will become clear shortly. We have not succeeded in solving (3.6) for any general form of the scalar function f. Special cases, however, can be solved explicitly, as we proceed to demonstrate. Assuming, for example, that f is a separable function of u and s, say $G(u)/F(s)$, Eq. (3.6) takes on the form of the usual BPE,[8]

$$\mathbf{t}_{u'}=\mathbf{t}\times\mathbf{t}_{s'}, \tag{3.7}$$

in terms of transformed variables $u'=\int^u G(u)du$ and $s'=\int^s F(s)ds$. This equation first appeared in the context of the nonlinear sigma model[8] and subsequently in magnetic systems for the case $h=K$, $s'=s$, and $u'=u$. It is known to support exact instanton[8] solutions and twist[9] solutions. More recently, it has also been found to support a hierarchy of multitwist[10] solutions.

For illustration, let us analyze two particular solutions of Eq. (3.7) for an open-ended curve: A single instanton and the single twist. The single instanton[8,9] of typical size λ centered at $(s_0',\ u_0')$ has the form

$$\begin{aligned} \cos\,\theta &= [(s'-s_0')^2+(u'-u_0')^2-\lambda^2]/[(s'-s_0')^2+(u'-u_0')^2+\lambda^2], \\ \phi &= \arctan(u'-u_0')/(s'-s_0'). \end{aligned} \tag{3.8}$$

A single twist[9] of width $1/k$ and velocity ω/k is given by

$$\cos\,\theta = \tanh(ks'-\omega u'), \quad \phi = (\omega s'+ku'). \tag{3.9}$$

Equations (3.8) and (3.9) give $\mathbf{t}(s,u)$ for the evolving instanton and twist curves, respectively. Since $\mathbf{t}=\partial\mathbf{r}/\partial s$, the curve profile at any instant of time u is obtained by integrating $\mathbf{t}$ with respect to s. The results of these integrations are given in Figs. 1 and 2.

It is instructive to derive explicitly the curvatures and torsions of the instanton (I) and the twist (T) curves from Eqs. (2.3), (3.8), and (3.9). A short calculation yields

$$\begin{aligned} K_I&=2\lambda/[(s'-s_0')^2+(u'-u_0')^2+\lambda^2], \\ \tau_I&=2(u'-u_0')/[(s'-s_0')^2+(u'-u_0')^2+\lambda^2], \end{aligned} \tag{3.10}$$

and

$$K_T=\sqrt{k^2+\omega^2}\ \mathrm{sech}(ks'-\omega u'), \quad \tau_T=\omega\ \tanh(ks'-\omega u'). \tag{3.11}$$

For the case $(h/K)=C=$constant, we recover the usual BPE, once again with a rescaled time, $u\rightarrow u'=Cu$. The dynamics of the curve are distinctly different for the instanton and the twist. The qualitative behavior of the curve for the two cases is obtained from Eqs. (3.10) and (3.11) as follows.

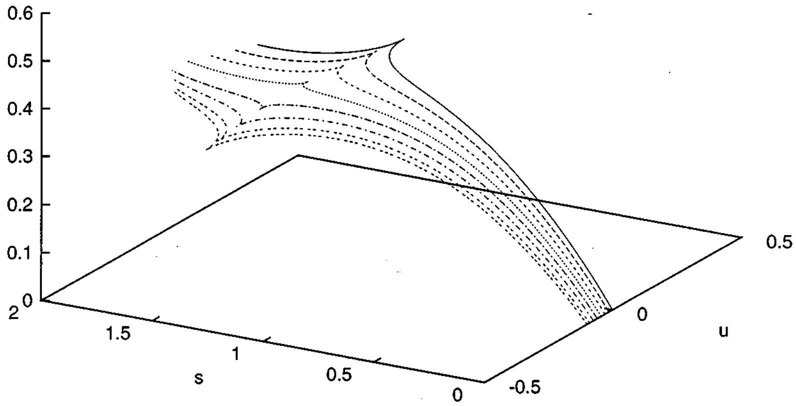

FIG. 1. The evolution of the one-instanton curve described by Eq. (3.8). The parameters are $s_0'=0.49$, $u_0'=0.1$, and $\lambda=1$.

For the instanton curve both the curvature and the torsion vanish as u tends to $\pm\infty$, and the curve is a straight line. At some intermediate time, the torsion is negative everywhere along the curve, increases as time goes on, vanishes at $u=Cu_0$, and turns positive thereafter. The curvature is always finite and reaches its maximum everywhere along the curve at Cu_0. In other words, an initially nonplanar curve with a given curvature becomes planar with a high curvature that goes as $1/[(s-s_0)^2+\lambda^2]$ at Cu_0 and then turns in the opposite nonplanar direction while its curvature decreases as the inverse square of the time. The single-twist solution describes a kink that moves along the curve with velocity ω/k. As u increases, the torsion changes from $-\omega$ to ω. The curvature vanishes exponentially fast at the end of the curve, while it reaches its maximal value $\sqrt{k^2+\omega^2}$ at $s=\omega k/u$. This qualitative description is borne out quantitatively in Figs. 1 and 2.

More general forms of F and G strongly modify the evolution of the curve. For example, consider an exotic case where $F(s)=A\sin s$ $(A>0)$ and $G(u)=1$, leading to $s'=A\cos s$ and $u'=u$. The one-instanton solution reads as

$$\begin{aligned} K_I&=2\lambda/[A^2(\cos s-\cos s_0)^2+(u-u_0)^2+\lambda^2],\\ \tau_I&=2(u-u_0)/[A^2(\cos s-\cos s_0)^2+(u-u_0)^2+\lambda^2]. \end{aligned} \tag{3.12}$$

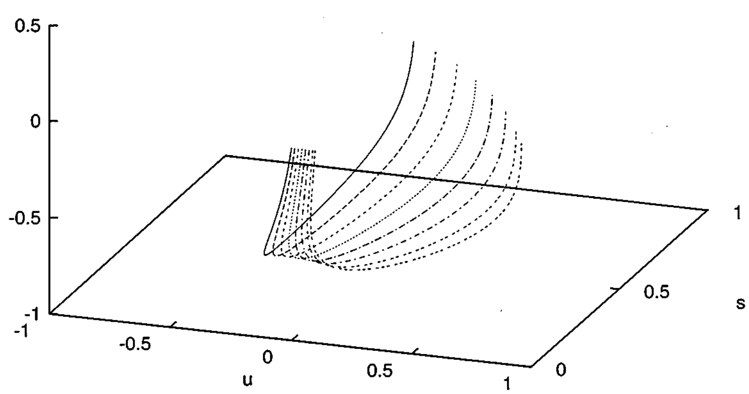

FIG. 2. The evolution of the one-twist curve described by Eq. (3.9). The parameters are $k=3$ and $\omega=2$.

At any given time the curvature and torsion oscillate with the position along the curve, the amplitude diminishing with time. The sign of $\tau_I \sim (u-u_0)$ persists throughout the curve, as before. Also as before, the curve approaches a straight line as $u \to \pm\infty$.

The single-twist solution for an open-ended curve becomes, in this instance,

$$K_T = \sqrt{k^2+\omega^2}\ \mathrm{sech}(Ak\cos s - \omega u),$$
$$\tau_T = \omega\ \tanh(Ak\cos s - \omega u). \tag{3.13}$$

For early times $u < -u_0 = -Ak/\omega$, there is no twist along the curve since the argument of the hyperbolic function has no zero. As time exceeds $-u_0$, a twist suddenly appears and is repeated periodically along the curve. The location of the twist within one period changes with time until at $u = u_0$ the twist disappears as suddenly as it appeared. Thus, this solution has a particle-like character, but in the *time domain*.

It must be borne in mind that the form of G/F depends on the physics that governs the dynamics of the curve. In other words, one still needs a physical argument to supply the equation of motion of the scalar function f.

IV. TRANSFORMATION OF GENERAL CURVE EVOLUTION TO THE MBPE

Having covered the above special cases, a natural question that arises is whether the general evolution can be reduced to the MBPE. We now proceed to show that this is indeed so, and that the evolution of *any arbitrary* curve can be described by this equation. We believe that this is a significant result, in that it reduces the evolution equation of the tangent vector to a relatively compact form, many of whose solutions are known exactly.

We start by seeking a particular transformation from the variables s and u, to a new coordinate system, $\xi(s,u)$ and $\eta(s,u)$:

$$\mathbf{t}_s = \mathbf{t}_\xi \xi_s + \mathbf{t}_\eta \eta_s\,; \quad \mathbf{t}_u = \mathbf{t}_\xi \xi_u + \mathbf{t}_\eta \eta_u\,. \tag{4.1}$$

Substituting (4.1) into (2.9) and simplifying, we get

$$j\mathbf{t}_\xi = [\xi_s\eta_s + (\alpha\xi_s - \beta\xi_u)(\alpha\eta_s - \beta\eta_u)](\mathbf{t}_\xi \times \mathbf{t}) + [\eta_s^2 + (\alpha\eta_s - \beta\eta_u)^2](\mathbf{t}_\eta \times \mathbf{t}) \tag{4.2}$$

and

$$j\mathbf{t}_\eta = [\xi_s^2 + (\alpha\xi_s - \beta\xi_u)^2](\mathbf{t} \times \mathbf{t}_\xi) + [\xi_s\eta_s + (\alpha\xi_s - \beta\xi_u)(\alpha\eta_s - \beta\eta_u)](\mathbf{t} \times \mathbf{t}_\eta), \tag{4.3}$$

where

$$j \equiv \xi_s\eta_u - \xi_u\eta_s = \frac{\partial(\xi,\eta)}{\partial(s,u)}$$

is the Jacobian of the transformation, which, for legitimacy, should not vanish. We now require that the transformation satisfies the condition

$$\xi_s\eta_s + \Gamma(\xi)\Gamma(\eta) = 0, \tag{4.4}$$

where $\Gamma(\mu) = (\alpha\mu_s - \beta\mu_u)$. Implementing this condition and using a little algebra, Eqs. (4.2) and (4.3) reduce to

$$\mathbf{t}_\eta = f_1(\mathbf{t} \times \mathbf{t}_\xi), \tag{4.5}$$

where the scalar function f_1 is given by

$$f_1 \equiv \sqrt{\frac{\xi_s^2 + \Gamma^2(\xi)}{\eta_s^2 + \Gamma^2(\eta)}}. \tag{4.6}$$

Note that with the above definition, the Jacobian is

$$j = [\eta_s \Gamma(\xi) - \xi_s \Gamma(\eta)].$$

Thus, with the class of transformations $(s,u) \rightarrow (\xi,\eta)$ that satisfy condition (4.4), the general curve evolution (2.8) is indeed reduced to the MBPE, Eq. (4.5). As in the old coordinate system, $\mathbf{t}_\eta$ and $\mathbf{t}_\xi$ are also orthogonal to $\mathbf{t}$ in the new coordinate system. This can be seen by writing $\mathbf{t} \cdot \mathbf{t}_s = \mathbf{t} \cdot \mathbf{t}_u = 0$ and expanding in terms of ξ and η. Alternatively, this follows from the fact that $\mathbf{t}$ is a unit vector in any coordinate system.

As mentioned earlier, in spite of the attractive compact form of Eq. (4.5) we are unable at present to solve this equation for an arbitrary form of f_1. However, if f_1 is *separable*, $f_1 = G_1(\eta)/F_1(\xi)$, with F_1 and G_1 some arbitrary integrable functions, then we can recast the evolution equation in the BPE form exactly as in case (ii) discussed above, by defining the following variables:

$$\xi'(s,u) = \int^{\xi} F_1(\xi) d\xi, \quad \eta'(s,u) = \int^{\eta} G_1(\eta) d\eta. \tag{4.7}$$

The resulting equation in the new variables has the BPE form

$$\mathbf{t}_{\eta'} = \mathbf{t} \times \mathbf{t}_{\xi'},$$

whose solutions have been discussed previously (see Sec. III). When translated to the original coordinates, these solutions can be written in terms of a new variable[10] $\psi = \frac{1}{2} \ln[(1-\cos\theta)/(1+\cos\theta)]$ as

$$\begin{aligned} \cos\theta &= \tanh\psi(\xi'(s,u), \eta'(s,u)), \\ \phi &= \phi(\xi'(s,u), \eta'(s,u)). \end{aligned} \tag{4.8}$$

ψ and ϕ can be shown[10] to satisfy Cauchy–Riemann relations in the $\xi'\eta'$ plane, so that they are harmonic functions of ξ' and η': ψ is the harmonic potential, while ϕ is the conjugate stream-function. Unlike in usual two-dimensional Laplacian problems, they can *diverge* anywhere (in particular at the boundaries) in the present context, without losing physical relevance. The reason is that the physical variable $\cos\theta = \tanh\psi$ continues to be finite, remaining in the allowed range $[-1,1]$, even if $\psi \rightarrow \pm\infty$. Thus solutions comprising polynomials of arbitrary degree are allowed[10] for open-ended curves. Similarly, the solutions for the angle ϕ can also diverge and yet remain physical, as it is only $\phi \bmod 2\pi$ that is relevant to the curve.

For a closed loop, the boundary conditions are periodic and the solutions must consist of periodic, harmonic functions in the variable s. The u variable need not be periodic, and if ξ' increases monotonically with u, the generic solutions for ψ and ϕ are combinations of oscillating harmonic modes with hyperbolic functions, e.g., $(\sin ks' \cosh ku')$. If, however, u' is a *periodic* function of u, we have an interesting situation where the terms are periodic in both s and u, corresponding to toroidal solutions.

V. APPLICATIONS

In this section, we discuss two physical applications of the curve evolution formalism developed above.

A. Local kinematics of an evolving space curve and its physical realizations

Interfaces, polymer chains, etc. are physical applications of space curves whose equations of motion can be either local or nonlocal. In many cases these equations take the form of a first-order ODE for the local velocity of the curve, namely,

$$\mathbf{r}_u = \mathbf{v}(x,y,z). \tag{5.1}$$

For instance, this can correspond to viscosity-dominated dynamics. We can write the right-hand side of this equation explicitly in terms of the local triad system ($\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$):

$$\mathbf{r}_u = U\mathbf{n} + V\mathbf{b} + W\mathbf{t}. \tag{5.2}$$

To relate the velocities U, V, W to the curvatures discussed above, we recall that $\mathbf{t}=\mathbf{r}_s$ and use the same compatibility conditions that led to Eqs. (2.4). These yield the following three relations:

$$0 = W_s - KU, \quad g = U_s - V\tau + WK, \quad h = V_s + U\tau. \tag{5.3}$$

We recall that only two of the ''curvatures'' K, τ, g, and h are independent [the rest are related through Eqs. (2.4)], and therefore these three relations give, in principle, the velocities U, V, W, say, in terms of the curvature K and torsion τ, and vice versa. Eliminating any two of the velocities in favor of the third yields a linear third-order ODE in s. For example, for case (ii) of Sec. III, with $g=0$ and $h=K$, we find after a short calculation that the velocities V and W satisfy, respectively,

$$\begin{gathered} [\{[(V_s - K)/\tau]_s + \tau V\}/K]_s + (V_s - K)(K/\tau) = 0, \\ \{[(W_s/K)_s + WK]/\tau\}_s + (\tau/K)W_s - K = 0. \end{gathered} \tag{5.4}$$

Although the above ODEs have complicated forms, it is clear that knowing the expressions for K and τ allows one to solve for V, W, and $U = W_s/K$, either analytically or numerically. We thus have the state of the curve at any time u. This establishes the connection between the kinematics of the physical curve, which can be observed and measured in the laboratory, and the quantities defined in the preceding analysis. This connection should prove useful in several applications, for instance, (a) when one has a physical model for the local motion motion of the curve (which must be recast in the form of moving curve equations to find the global dynamics), or (b) when one might wish to obtain curves of particular shapes as a function of time, for example, to engineer a specific type of linking of a biological molecular chain or a polymer.

As an example of the above analysis, consider again the conditions of case (ii) above, $g=0$ and $h=K$. For clarity, let us focus on the single twist solution [Eq. (3.11)] with $k=0$ and $\omega \neq 0$:

$$K_T = \omega \text{ sech } \omega u, \quad \tau_T = \omega \tanh \omega u. \tag{5.5}$$

The curvature and the torsion are *independent* of s and take on a particular value for a given u. At $u=-\infty$, $K_T=0$ and the curve profile is a straight line along the z axis. For finite $u<0$, we have $K_T(>0)$ and $\tau_T(<0)$ constant along the curve for each fixed u, giving a helical curve. This profile gets flattened to a circle as $u\to 0$ ($K_T=\omega$, $\tau_T=0$). It then unwinds in the opposite direction for increasing $u>0$, with $K_T>0$, $\tau_T>0$, and finally points along the $-z$ axis for $u\to\infty$. Thus, the special case $k=0$ simulates the unwinding and straightening of a curve (a polymer, say) that was initially coiled up in a circle, at $u=0$. The ODE satisfied by U in this case is simple to write down from Eqs. (5.3), since $\tau_s = K_s = 0$ and $(K^2+\tau^2)=\omega^2$. It is

$$U_{ss} + \omega^2 U = K(u)\tau(u), \tag{5.6}$$

which, with the given boundary conditions, can then be integrated to yield $U(s,u)$ as a function of ω, K, and τ. We find

$$U(s,u)=(U_0-V_0/\omega^2)\cos(\omega s)+(V_0/\omega^2)\sin(\omega s)+[K(u)\tau(u)/\omega^2], \tag{5.7}$$

where $U_0=U(s=0,\ u)$ and $V_0=U_s(s=0,\ u)$ are the boundary conditions at the instant u. The explicit local kinematics of the unwinding and straightening of a helical polymer chain would be useful in several biological applications.

B. The inhomogeneous antiferromagnetic chain

The spin evolution equation for the homogeneous case[9] has been derived in earlier work in another context. Here, we wish to concentrate on the new insight that the present analysis provides, for the inhomogeneous case.

Consider the classical inhomogeneous antiferromagnetic chain described by the Hamiltonian

$$H=-\sum_i J_i \mathbf{S_i}.\mathbf{S_{i+1}}, \tag{5.8}$$

where $\mathbf{S}_i=(S_i^x,S_i^y,S_i^z)$ represents the classical spin vector of constant magnitude S at the site i and J_i is the site-dependent exchange interaction. The equation of motion for $\mathbf{S}_i$ is found by using the Poisson bracket relation $dS_l^\delta/dt=\{S_l^\delta,H\}=\Sigma_i\epsilon_{\alpha\beta\gamma}(\partial S_l^\delta/\partial S_i^\alpha)(\partial H/\partial S_i^\beta)S_i^\gamma$. Since $J_i<\ 0$ for all i, the nearest neighbor spin vectors will tend to align in antiparallel directions for low energies. Hence, it is convenient to study the problem by writing down the evolution of the spin vector $\mathbf{S}_{2i}$ at an even site and $\mathbf{S}_{2i-1}$ at an odd site as follows:

$$\begin{aligned} d\mathbf{S}_{2i}/dt&=\mathbf{S}_{2i}\times(J_{2i}\mathbf{S}_{2i+1}+J_{2i-1}\mathbf{S}_{2i-1}),\\ d\mathbf{S}_{2i-1}/dt&=\mathbf{S}_{2i-1}\times(J_{2i-1}\mathbf{S}_{2i}+J_{2i-2}\mathbf{S}_{2i-2}). \end{aligned} \tag{5.9}$$

In the continuum approximation, $\mathbf{S}_{2i}\to\mathbf{S}_e(x)$, $\mathbf{S}_{2i-1}\to\mathbf{S}_o(x-a)$; $J_{2i-n}\to J(x-na)$, $n=0$, 1, 2, where a is the nearest neighbor separation. Note that the Taylor expansion parameter is $2a$ for the sublattice spin vectors (because of nearest neighbor antiparallelism) and a for the interaction J, which is assumed to vary smoothly along the chain to allow for the continuum approximation. Thus, using

$$\mathbf{S}_{2i+1}\to\mathbf{S}_o(x-a)+2a(\partial\mathbf{S}_o/\partial x),\quad \mathbf{S}_{2i-2}\to\mathbf{S}_e(x)-2a(\partial\mathbf{S}_e/\partial x)$$

and

$$J_{2i-n}\to J(x)-na(\partial J/\partial x),$$

in Eqs. (5.9), we get

$$\begin{aligned} \mathbf{S}_{e,t}&=\mathbf{S}_e\times[J(x)(\mathbf{S}_0+2a\mathbf{S}_{o,x})+(J(x)-aJ_x)\mathbf{S}_o],\\ \mathbf{S}_{o,t}&=\mathbf{S}_o\times[(J(x)-2aJ_x)\mathbf{S}_e+(J(x)-2aJ_x)(\mathbf{S}_e-2a\mathbf{S}_{e,x})], \end{aligned} \tag{5.10}$$

where the subscripts x and t stand for partial derivatives. When $J(x)=J=\text{const}$, these reduce to the equations for the *homogeneous* antiferromagnetic chain.[9] For the inhomogeneous case, we define two vectors $\mathbf{P}$ and $\mathbf{Q}$ as follows:

$$\begin{aligned}[\mathbf{S}_e(x)-\mathbf{S}_o(x-a)]&=2S\mathbf{P},\\ [\mathbf{S}_e(x)+\mathbf{S}_o(x-a)]&=2S\mathbf{Q}.\end{aligned} \tag{5.11}$$

Combining Eqs. (5.10) and using the definitions (5.11), a short calculation yields

$$\begin{aligned}\mathbf{Q}_t&=2J(x)Sa(\mathbf{P}\times\mathbf{Q})_x-2aSJ_x(\mathbf{P}\times\mathbf{Q}),\\ \mathbf{P}_t&=4S(J(x)-aJ_x)(\mathbf{P}\times\mathbf{Q})+2J(x)Sa[(\mathbf{Q}\times\mathbf{Q}_x)-(\mathbf{P}\times\mathbf{P}_x)].\end{aligned} \tag{5.12}$$

For the antiferromagnetic chain, it is clear that for low energies, $|\mathbf{Q}|$ is much smaller than $|\mathbf{P}|$. Furthermore, note that $\mathbf{Q}=0$ is a possible exact solution for Eq. (5.12). In this case, Eq. (5.12) simplifies considerably and does not contain derivatives of $J(x)$ anymore. This solution represents dimer-like locked spin pairs along the chain, with $\mathbf{P}=\mathbf{S}_e/S$ becoming a unit vector in this limit. Rescaling variables $x/2a\rightarrow s$, $St\rightarrow u$, and $\mathbf{P}\rightarrow\mathbf{t}$, the second of Eqs. (5.12) becomes

$$\mathbf{t}_u=J(s)(\mathbf{t}\times\mathbf{t}_s). \tag{5.13}$$

This equation has the form of the MBPE [Eq. (3.6)]. Defining a new variable $s'=\int^s ds/J(s)$ and $u'=u$, this takes on the simpler BPE form [Eq. (3.7)]. The solutions for the polar angle θ and the azimuthal angle ϕ of $\mathbf{t}$ are given in Eqs. (3.8) for the instanton class, and in Eqs. (3.9) for the twist class. Thus, we see that the above formalism and solutions apply to the *inhomogeneous* antiferromagnetic spin chain, the requirement being that $1/J(s)$ is an integrable function.

VI. CONCLUDING REMARKS

Moving space curves can be represented by two sets of Frenet–Serret equations that describe the spatial (s) and temporal (u) evolution of the vector triad ($\mathbf{t}$,$\mathbf{n}$,$\mathbf{b}$), by specifying the curvatures K, g, and h and the torsions τ and τ_0. All these quantities are, in general, functions of s and u. If compatibility conditions are imposed on the vectors, then coupled nonlinear partial differential equations for the curvatures and the torsions are obtained. For certain special choices of curve evolution, it is possible to obtain[3] certain well-known completely integrable equations with soliton solutions. While such a result implies that the underlying evolution of $\mathbf{t}$ is also completely integrable, finding its explicit solution is a nontrivial task, in practice. In this paper we have studied the problem of moving curves from a different angle and demonstrated that for a fairly large class, the evolution of $\mathbf{t}$ can be reduced to a solvable form, enabling us to write down its solution explicitly.

We have used the following strategy: We start by casting the general evolution of $\mathbf{t}$ in the form $K\mathbf{t}_u=[g\mathbf{t}_s-h(\mathbf{t}_s\times\mathbf{t})]$ [Eq. (2.8)]. Using this, first we show that if $h=0$ and (g/K) is a separable function of s and u, $\mathbf{t}$ satisfies a *linear* solvable equation with unidirectional traveling wave solutions. Next, we show that if $g=0$ and (h/K) is a separable function, $\mathbf{t}$ satisfies the well-known BPE equation, which is a *nonlinear* exactly solvable equation. In both these cases, the corresponding equations are in terms of transformed variables, which are functions of s and u. It is interesting that we are able to generalize this latter result to the case when *both* g and h are nonvanishing, to obtain once again a BPE in terms of appropriate transformed variables. Our results demonstrate how a large class of curve evolutions with appropriate curvatures and torsions can be effectively mapped to the BPE. This equation supports exact instanton and twist solutions. If Figs. 1 and 2, we have displayed the curve evolutions corresponding to the single instanton and the single twist. Multi-instanton solutions are well known. We have recently found multi-twist[10] solutions for the BPE. These and the associated curve evolutions will be reported elsewhere. Application to the kinematics of curve motion shows interesting results for the time evolution of the local velocity components on the curve. As a second application, we find that the dynamics of the inhomogeneous antiferromagnetic chain can be mapped to a BPE curve evolution in terms of a transformed spatial variable, which can be expressed in terms of the interaction between the spins on the chain.

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