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On the twist excitations in a classical anisotropic antiferromagnetic chain

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Abstract

In a recent Erratum [M. Daniel and R. Amuda, Phys. Lett. A 224 (1997) 389] the authors have explained that an algebraic error in their earlier analysis [Phys. Lett. A 191 (1994) 46] of a nonlinear evolution equation for a certain anisotropic antiferromagnetic chain in a magnetic field invalidates the twist solution obtained by them for any finite anisotropy. Using a transformed variable and a different analysis of that evolution equation in the presence of a field, we obtain a $\pi/2$-traveling twist solution for a finite anisotropy and a π-twist for negligible anisotropies. © 1997 Elsevier Science B.V.

Nonlinear dynamics of magnetic chains with various symmetries is a topic that has attracted a lot of attention in recent years [1]. The classical isotropic ferromagnetic (FM) chain described by the Hamiltonian $H = -J \sum_n \mathbf{S}_n \cdot \mathbf{S}_{n+1}$, $J > 0$, $(S_n)^2 = S^2 =$ const, is well known to be a completely integrable system in the continuum limit and supports nonlinear excitations which are pulse-type, non-topological soliton configurations [2]. In contrast, the study of the nonlinear dynamics of the isotropic antiferromagnetic (AFM) chain, $J < 0$, is more complicated, essentially because the nearest neighbour spin vectors tend to be antiparallel for low energies. Thus the evolution equations of spin vectors at the odd and even sites, i.e. the two sublattices, must be analysed individually. In view of this, the continuum equations for the AFM chain can be derived in many ways.

The sublattice vector formulation of Baryakhtar and Ivanov [3] derives the continuum coupled equations of motion for the staggered magnetization $\boldsymbol{\eta}(x, t) = (\boldsymbol{S}_e - \boldsymbol{S}_o)/2S$ and the total magnetization $\boldsymbol{\zeta}(x, t) = (\boldsymbol{S}_e + \boldsymbol{S}_o)/2S$ (where $\boldsymbol{S}_e$ and $\boldsymbol{S}_o$ represent spin vectors at even and odd sites), starting from the continuum version of the AFM Hamiltonian expressed in terms of these dynamical variables. On neglecting certain terms in the Hamiltonian, one can obtain [1] a Lorentz-invariant nonlinear sigma model with the well known antiferromagnetic magnons as the low-energy excitations for the isotropic case. More recently, in an alternative sublattice vector formulation [4] one starts with the discrete equations of motion for the spin vectors at the odd and even sites and derives the continuum equation of motion for $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ from them. It should also be noted that although the *total* z component of $\boldsymbol{S}$ is conserved for the discrete Hamiltonian (as in the FM case), that on a sublattice is not. This suggests that a domain-wall

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like topological configuration in which the two sublattice spin vectors exchange roles can arise as a nonlinear excitation, and this was found in Ref. [1] for the isotropic case in the absence of a magnetic field.

Applying the latter formulation [4] to an anisotropic chain with an easy-axis anisotropy $A\Sigma_l S_l^{z2}$, the following continuum equation of motion for $\boldsymbol{\eta}$ was derived some time ago by Daniel and Amuda [5] in the limit of vanishing $\boldsymbol{\zeta}$:

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = -\boldsymbol{\eta}\times\left[\boldsymbol{\eta}_x + \left(\tilde{A}\eta^z - \tilde{B}\right)\boldsymbol{z}\right]. \tag{1}$$

Here $\boldsymbol{\eta}$ is a unit vector. The quantities $\tilde{A}$ and $\tilde{B}$ represent respectively, the strength of the anisotropy and the magnetic field (in the z-direction) in units of J. For ready reference, note that Eq. (1) is Eq. (3.3a) of Ref. [5], rewritten in our notation. Here, we wish to point out that this equation is actually valid only for a staggered anisotropy $A\Sigma_l(-1)^l S_l^{z2}$ rather than an easy-axis one. It can be easily verified by an inspection of Eq. (2.3) in Ref. [5] that an easy-axis would lead to the *same* sign in front of A in Eqs. (2.4a) and (2.4b), whereas the authors obtain *opposite* signs. Thus the only case which would lead to opposite signs (which is necessary to obtain Eq. (1)) without tampering with the basic spin evolution equations is that of a staggered anisotropy. They then proceeded to solve Eq. (1) using complex analysis and obtained a twist-like solution (their Eq. (3.11)). However they have recently explained in their Erratum [6] that their analysis was inconsistent due to an algebraic error which invalidates that solution for any finite anisotropy. In what follows, we analyse Eq. (1) using a different approach and obtain a $\pi/2$-twist solution for this case. We also show that in the limit of vanishing $\tilde{A}$, a π-twist is obtained, even in the presence of a field. [1]

Since $\boldsymbol{\eta}$ is a unit vector, it can be represented in terms of spherical polar coordinates by

$$\boldsymbol{\eta} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{2}$$

[1] A π-twist in the *absence* of a field for the *isotropic* case was found in Ref. [4].

Using Eq. (2) in Eq. (1) yields

$$\theta_x/\sin\theta = -\varphi_t + (\tilde{A}\cos\theta - \tilde{B}), \tag{3a}$$

$$\theta_t/\sin\theta = +\varphi_x. \tag{3b}$$

We define a new variable ψ through the relation

$$\cos\theta = \tanh\psi \tag{4a}$$

and set

$$\tilde{\varphi} = \varphi + \tilde{B}t. \tag{4b}$$

Eqs. (3) yield

$$\psi_x = \tilde{\varphi}_t - \tilde{A}\tanh\psi, \tag{5a}$$

$$\psi_t = -\tilde{\varphi}_x. \tag{5b}$$

Thus we see that in these equations, the external magnetic field is gauged out and merely gives rise to a linear increase of the angle φ with time. Looking for traveling wave solutions of the form $\psi(\xi)$ and $\tilde{\varphi}(\xi)$, where $\xi = (x - vt)$ with $v = \text{const}$, we get

$$\psi_\xi = -v\tilde{\varphi}_\xi - \tilde{A}\tanh\psi, \tag{6a}$$

$$v\psi_\xi = \tilde{\varphi}_\xi. \tag{6b}$$

Eq. (6b) is readily integrated to yield

$$\tilde{\varphi} = v\psi + \tilde{\varphi}_0, \tag{7}$$

where φ_0 is a constant. Thus Eq. (6a) can also be integrated and gives

$$\psi = \ln\left\{C_0 e^{-\xi/l} + \left(C_0^2 e^{-2\xi/l} + 1\right)^{1/2}\right\}, \tag{8}$$

where $l = (1 + v^2)/\tilde{A} \neq 0$ and C_0 is a constant to be fixed by the initial condition. From Eqs. (4b), (7) and (8),

$$\varphi = \tilde{\varphi}_0 - \tilde{B}t + v\ln\left\{C_0 e^{-\xi/l} + \left(C_0^2 e^{-2\xi/l} + 1\right)^{1/2}\right\}. \tag{9}$$

The solution for $\boldsymbol{\eta}$ is found from Eqs. (2) and (4a):

$$\boldsymbol{\eta}(x, t) = \operatorname{sech}\psi\{\cos\varphi\,\boldsymbol{x} + \sin\varphi\,\boldsymbol{y}\} + \tanh\psi\,\boldsymbol{z} \tag{10}$$

with ψ and φ as in Eqs. (8) and (9). Eq. (8) shows that $\psi \to 0$ when $\xi \to \infty$ and $\psi \to \infty$ when $\xi \to -\infty$. Using this in Eq. (10) yields the result that $\boldsymbol{\eta}$ is a $\pi/2$-twist, for all $\tilde{B}$, i.e. a configuration with the staggered magnetization $\boldsymbol{\eta} \to \boldsymbol{z}$ as $x \to -\infty$ and $\boldsymbol{\eta} \to (\sin\varphi\,\boldsymbol{x} + \cos\varphi\,\boldsymbol{y})$ as $x \to +\infty$. Note also that both $\psi(x, t)$ and $\tilde{\varphi}(x, t)$ are functions of $(x - vt)$ with the *same* traveling wave velocity v, i.e. $V_\psi = V_{\tilde{\varphi}} = v$. The quantity $l = (1 + v^2)/A$ sets the scale of the solution. As the anisotropy A vanishes, this scale diverges, implying that $\psi=$ const along the chain. In this limit, however, the equation of motion for $\boldsymbol{\eta}$ reduces to

$$\frac{\partial\boldsymbol{\eta}}{\partial t}=-\boldsymbol{\eta}\times\left(-\boldsymbol{\eta}_x+\tilde{B}\boldsymbol{z}\right), \tag{11}$$

which leads to (see Eqs. (5))

$$\psi_x=\tilde{\varphi}_t, \quad \psi_t=-\tilde{\varphi}_x. \tag{12}$$

These support the (simplest) solutions

$$\psi=(kx-\omega t),$$

$$\tilde{\varphi}=(\omega x+kt), \tag{13}$$

where k and ω are arbitrary constants. From Eq. (4b) $\varphi=\tilde{\varphi}-\tilde{B}t=\omega x+(k-\tilde{B})t$. Using these in Eq. (10), we get

$$\boldsymbol{\eta}=\operatorname{sech}(kx-\omega t)\left\{\cos\left[\omega x+(k-\tilde{B})t\right]\boldsymbol{x}+\sin\left[\omega x+(k-\tilde{B})t\right]\boldsymbol{y}\right\}+\tanh(kx-\omega t)\boldsymbol{z}. \tag{14}$$

This is a π-twist, i.e. $\boldsymbol{\eta}\to\pm\boldsymbol{z}$ as $x\to\pm\infty$. From Eq. (13) we see that the twist itself travels along the chain with a velocity $V_\psi=\omega/k$, whereas within the width of this traveling twist, the velocity V_φ of the periodic wave for the projection of $\boldsymbol{\eta}$ in the x–y plane is $V_\varphi=(\tilde{B}-k)/\omega$. A pure traveling wave solution for $\boldsymbol{\eta}$ requires $V_\psi=V_\varphi$, which is satisfied when $\omega^2=k(\tilde{B}-k)$. Since ω and k are real, this in turn can be satisfied only when the effective magnetic field $\tilde{B}$ and the wave-vector k are such that $(\tilde{B}-k)$ and k have the same sign, otherwise $V_\psi\neq V_\varphi$. In the zero-field case $\tilde{B}=0$, it is evident that the two waves *must* have different velocities.

It is interesting to note that for the case $\tilde{A}=0$, when a π-twist is obtained for Eq. (1), ψ and $\tilde{\varphi}$ are linear in x and t (Eq. (13)). In contrast, for $\tilde{A}\neq 0$, when a $\pi/2$-twist is obtained, ψ and φ are nonlinear functions of $(x-vt)$ (see Eqs. (8) and (9)). The width of this twist depends on A/J, but when $A=0$, there is an *abrupt* transition from a $\pi/2$-twist to a π-twist solution, suggestive of a bifucation-like behaviour as A is switched on or off.

Having obtained the traveling twists using the moving coordinate approach, let us investigate what happens when complex analysis is carried out using the approach suggested in Ref. [5]. As mentioned in Ref. [6], that approach fails for any *finite* anisotropy. For the case of *negligible* anisotropy,

$$\theta_z/\sin\theta=(\mathrm{i}\varphi_z-C), \tag{15a}$$

$$\theta_{z^*}/\sin\theta=-(\mathrm{i}\varphi_{z^*}-C^*) \tag{15b}$$

where $z=t+\mathrm{i}x$ and $C^*=\mathrm{i}\tilde{B}/2$. (Note that Eqs. (15) are Eq. (3.7a) and Eq. (3.7b) of Ref. [6].) Integrating Eqs. (15) yields [5]

$$\tan(\theta/2)\exp(-\mathrm{i}\varphi+Cz)=\Omega(z^*) \tag{16}$$

and its complex conjugate. Here, $\Omega(z^*)$ is an arbitrary function of z^*.

Following the steps in Ref. [5], one obtains the condition that $(\mathrm{i}\varphi+C^*z^*)=f$ where f is a real function. On substituting for C^* and z^* and equating real and imaginary parts of this equation, we get

$$\varphi=-\tilde{B}t/2, \quad f=\tilde{B}x/2.$$

The choice $\Omega=1$ [5] in Eq. (16) will then yield

$$\cos\theta=\tanh f. \tag{18}$$

Using Eqs. (17) and (18) in Eq. (2) we obtain the following solution for the staggered magnetization:

$$\boldsymbol{\eta}(x,t)=\operatorname{sech}f\{\cos\varphi\,\boldsymbol{x}+\sin\varphi\,\boldsymbol{y}\}+\tanh f\,\boldsymbol{z}. \tag{19}$$

This is *different* from the solution (Eq. (3.11)) written down in Ref. [5] and referred to as valid in Ref. [6] since $\varphi\neq f$ in Eq. (19). Further, the expressions for f and φ given in Eq. (17) show that the solution obtained with this type of complex analysis is a non-traveling twist with the spin vectors processing with a constant frequency $\tilde{B}/2$ at all sites along the chain, and is distinct from the traveling twist Eq. (14).

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