PHYSICAL REVIEW E VOLUME 51, NUMBER 4 APRIL 1995

Two-dimensional Laplacian growth as a system of creating and annihilating particles

Raphael Blumenfeld[1,*] and Robin C. Ball[2]
[1]Princeton Materials Institute, Princeton University, Bowen Hall, 70 Prospect Avenue, Princeton, New Jersey 08540-5211
[2]Cavendish Laboratory, Cambridge University, Madingley Road, Cambridge CB3 0HE, United Kingdom
(Received 4 November 1994)

We study the problem of an interface growing in a two-dimensional Laplacian field in terms of the statistics of the distribution of interacting particles. This equivalent many-body system consists of two species of particles that interact, Z and P, confined to a disk. We show that by tip splitting, the system reduces its surface energy. Tip splitting along the physical interface corresponds, in the equivalent many-body system, to particle production of the form $Z \rightarrow 2Z + P$ and we therefore introduce a constitutive rule that facilitates this mechanism. We discuss the relation between the spatial distribution of the reacting particles and the evolution of the statistics of the surface. In particular, we derive the multifractal function of diffusion-limited aggregation in terms of this spatial distribution.

PACS number(s): 68.70.+w, 81.10.Dn, 11.30.Na, 81.30.Fb

I. INTRODUCTION

Laplacian growth processes abound in nature and have been the focus of much attention recently. Few examples are: electrical discharge, solidification, dendritic growth, viscous fingering, chemical dissolution, and evolution of bacterial colonies. Much effort has been directed towards theories that predict the rich variety of patterns that such processes lead to. One of the simplest problems to formulate in this context is the process of diffusion-limited aggregation (DLA) in two dimensions [1]. Despite its simplicity and the existence of a huge body of phenomenological examples and computer simulations, a fundamental theory for this process is yet to emerge. Although a few probabilistic and renormalization group approaches have been put forward [2–4] there is still no clear understanding how, starting from the local equations of motion (EOM), to predict the asymptotic morphology of the growing pattern. In fact, the number of theoretical results in the literature on DLA is surprisingly small *vis-á-vis* the large amount of phenomenological knowledge on the subject. One of the key observations in DLA is that the distribution of the electrostatic field (or concentration gradient) along the growing surface flows to a stable limit, sometimes expressed in terms of a universal function, $f(\alpha)$ [5]. This indicates that the asymptotic morphology of this process is robust against fluctuations in initial conditions and that, in principle, the asymptotic behavior is determined to a large extent by the local growth mechanism. In this paper we take some steps towards understanding this growth process.

The mathematical problem can be posed as follows: Consider a simply connected region in a two-dimensional plane ζ outside of which there exists a field Φ that obeys Laplace's equation

$$\nabla^2\Phi = 0 . \tag{1.1}$$

On the boundary of the region (the interface between the growing region and the rest of the plane) and on a circular boundary far away from it, the field assumes constant values, Φ_1 and Φ_2, respectively, with $\Phi_1 < \Phi_2$. The interface moves outwards according to a local constitutive relation

$$v_n(s) = -\nabla\Phi|_s . \tag{1.2}$$

It has been proposed [6,7] to conformally map the interior of the physical interface onto the unit disk and study the evolution of the map instead of that of the actual interface. Several workers applied this idea recently to derive results for particular systems [8,9]. An attractive feature of this approach is that since the mapping function must be analytic only exterior to the interface, it can possess zeros and poles inside the unit disk. Thus the problem can be formulated in terms of the dynamics of these singularities, and indeed most of the studies in this problem specialized on particularly simplified sets of initial singularities and their time evolution. It has been found [7,9] that generically the singularities of the map propagate towards the unit circle to eventually hit it after a *finite* time. When this happens a cusp forms along the physical interface, the map ceases to be analytic outside the unit circle, and the formalism breaks down. Therefore the mathematical problem is ill posed unless surface effects are introduced to prevent this catastrophe.

In this paper we use this formalism to analyze the evolution of the statistics of the interface. The idea that underlies our approach is the following: There is a one-to-one correspondence between the structure of the evolving interface and the spatial distribution of the singularities in the unit disk. Therefore, by studying the statistics of the latter, we expect to learn about the asymptotic pattern of the growing physical surface. Specifically, we want to relate the distribution of the potential field (or, rather its gradient) immediately in front of the interface to the distribution of the location of the singularities of the map. As mentioned, we are motivated by the

*Present address: CNLS, MS B258, Los Alamos National Laboratory, Los Alamos, NM 87545.
Electronic address: RAFI@GOSHAWK.LANL.GOV

numerous observations that the statistics of the moving boundary converge to a stable limit form, usually fractal. One manifestation of this phenomenon is the appearance of a well-defined multifractal function [5] $f(\alpha)$ that is independent of time or length scale (e.g., in DLA) [10,11]. We therefore expect the distribution of singularities to reflect this asymptotic behavior by also converging to a stable limiting form.

We present here several results, first regarding the initial motion of the singularities inside the unit circle without surface effects. These include the following: (i) the time dependence of the trajectory of a zero close to the unit circle in a dilute approximation; (ii) numerical solutions for trajectories of singularities for several random initial positions; (iii) the time dependence of the increase in curvature as a zero approaches the unit circle. In the second part of the paper we introduce an interface-curvature-dependent mechanism that prevents formation of cusps as follows: When a zero Z gets too close to the unit circle it "spawns" a new zero-pole pair, $Z \rightarrow 2Z+P$. By construction, this mechanism satisfies several necessary constraints on the map (to be discussed below). The spawning is triggered by a threshold value of the local curvature just in front of the approaching zero, implying a field interpretation [12], where the surface energy induces a field in which the zeros and poles move. The EOM are then integrated from the new configuration. We present analytical results concerning the behavior of the singularities after the spawning event. The trajectories and the relation to tip splitting in the physical process are discussed. In the third part of the paper, we study the relations between the distribution of the singularities of the map within the unit disk and the growth probability distribution along the physical interface. In particular, we derive the multifractal spectrum, $f(\alpha)$ in terms of the distribution of the singularities.

II. FORMULATION OF THE PROBLEM

We start from the EOM of the boundary in the form of Shraiman and Bensimon [7]

$$\partial_t \gamma = -i\partial_s \gamma \{ |\partial_s \gamma|^{-2} + ig(s) \} \equiv -i\partial_s \gamma G(s) , \quad (2.1)$$

where t stands for time, s is a parameter that runs along the boundary and $g(s)$ is a real function. This equation is the limit of a map $\zeta = F(z,t) \equiv F(z)$ that takes the unit circle in the z plane to the physical boundary in the ζ plane (which, for simplicity, is assumed to be simply connected),

$$\gamma(s) = \lim_{z \rightarrow e^{is}} F(z) .$$

The term $\mathrm{Re}\{G\} = |\partial_s \gamma|^{-2}$ in (2.1) represents propagation in the normal direction and originates directly from the constitutive relation (1.2). The second term $\mathrm{Im}\{G\}$ represents a tangential velocity that causes "sliding" of a point s along the interface. This term is necessary to ensure that the right-hand side of (2.1) is the limit of an analytic map (a different way of regarding it is that this term ensures that the parametrization remains the same as time goes on), and it is constructed by using Cauchy-Riemann relations (see details below). In the following it is more convenient to work with $F'(z) = dF/dz$. Noting that as z approaches the unit circle

$$-i\partial_s \gamma = zF'(z) , \quad (2.2)$$

and replacing $G(s)$ in (2.1) by an analytic function $G(z)$ whose real part tends to $|\partial_s \gamma|^{-2}$ as $z \rightarrow e^{is}$, we can rewrite the EOM in the form

$$\partial_t (\ln F') = \frac{1}{F'} \frac{d}{dz} [zF'(z) G(z)] . \quad (2.3)$$

To make progress, several workers [7,8] focused on specific maps and particular initial conditions. For the purpose of the present treatment it is necessary that we discuss the general case and arbitrary initial conditions. A significant point to note at this stage is that the map is already constrained by the very formulation of the problem: First, the map and its inverse must be analytic in z outside the unit circle, excluding the point at infinity. This means that the poles and zeros of $F'(z)$ must be confined to the unit disk. Second, the map should keep the boundary conditions far away from the growing surface unchanged. Therefore the topology far away from the growth must remain invariant under the map, leading to $\lim_{z \rightarrow \infty} F' = A$, where A is uniform over the plane (i.e., independent of spatial coordinates but it may be time dependent). A third necessary constraint will be discussed later. To accommodate the above two constraints, we choose the map to have the general form $F'(z) = \Pi_1(z)/\Pi_2(z)$, where $\Pi_1(z)$ and $\Pi_2(z)$ are polynomials of the *same* degree

$$F'(z) = A(t) \prod_{n=1}^{N} \frac{(z - Z_n)}{(z - P_n)} , \quad (2.4)$$

where $\{Z\}$ are the zeros of Π_1 and $\{P\}$ are the zeros of Π_2. This form is sufficiently general for the above constraints since it can be shown that it can approximate any given curve arbitrarily accurately by choosing N as large as desirable [13]

The next step is to find the form of the analytic function $G(z)$. For $|z| > 1$, this is done by carrying out a Hilbert transform [7]

$$G(z) = \frac{1}{2\pi i} \oint \frac{d\omega}{\omega} \frac{\omega + z}{\omega - z} \mathrm{Re}\{G(\omega)\} , \quad (2.5)$$

where the integration path is around the unit circle and where, using (2.2), we have $\mathrm{Re}[G(z)] = |zF'(z)|^{-2}$. Observing that $z^* = 1/z$ on the unit circle we obtain

$$G(z) = \frac{1}{A^2(t)} \left\{ G_0 + \sum_{n=1}^{N} Q(Z_n)/(z - Z_n) \right\} , \quad (2.6)$$

where $Q(Z_n)$ is the residue of $|F'(z)|^{-2}$:

$$Q(Z_n) = \frac{2\Pi_2^*(1/Z_n)\Pi_2(Z_n)}{\Pi_1^*(1/Z_n) \prod_{m' \neq n}^{N} (Z_n - Z_{m'})} ,$$

$$G_0 = \sum_{n=1}^{N} \frac{Q(Z_n)}{2Z_n} - \frac{\Pi_2(0)}{\Pi_1(0)} .$$

Substituting this expression into the EOM of the surface and contour integrating around each of the singularities on both sides yields their EOM

$$
\begin{aligned}
-\dot{Z}_n &= \frac{Z_n}{A^2(t)}\left\{G_0+\sum_{m'\neq n}\frac{Q(Z_n)+Q(Z_{m'})}{Z_n-Z_{m'}}\right\} \\
&\quad+\frac{Q(Z_n)}{A^2(t)}\left\{1-\sum_m\frac{Z_n}{Z_n-P_m}\right\}, \\
-\dot{P}_n &= \frac{P_n}{A^2(t)}\left\{G_0+\sum_m\frac{Q(Z_m)}{P_n-Z_m}\right\}, \\
\dot{A}(t) &= A(t)G_0 .
\end{aligned} \tag{2.7}
$$

As mentioned, these equations are subject to the aforementioned two constraints, and we now introduce the third constraint on the map: We require that F has no branch cuts as $z\to\infty$, namely,

$$\frac{1}{2\pi i}\oint F(z)dz=0 . \tag{2.8}$$

This leads to the following sum rule:

$$\sum_n R_n=\sum_n(P_n-Z_n)\prod_{m'\neq n}\frac{P_n-Z_{m'}}{P_n-P_{m'}}=0 , \tag{2.9}$$

where R_n are the residues of F' at its poles P_n. It is possible to show that this sum rule is equivalent to the requirement that

$$\sum_n P_n=\sum_n Z_n . \tag{2.10}$$

Namely, assigning a dipole to the quantity Z_n-P_n [the pairs (Z_n,P_n) can be assigned arbitrarily], the requirement is that the total dipole vanishes. By subtracting the two equations (2.7) from each other and summing over n it can be demonstrated that this constraint is satisfied identically by the EOM. Note that if we regard the zeros and poles as oppositely charged particles then the system is constrained to be both charge and dipole neutral. In passing we also comment that the residues R_n are important quantities in this formulation because (i) the map F can be explicitly expressed in terms of these quantities,

$$\zeta=F(z)=A(t)\left[z+\sum_{n=1}^{N}R_n\ln(z-P_n)\right] ; \tag{2.11}$$

(ii) the time evolution of these quantities follows [14]:

$$R_n(t)A(t)=R_n(0)A(0)=(\text{constant of the motion}) , \tag{2.12}$$

where the time-dependent scaling $A(t)$ is the same for all n and is exactly the rescaling factor in (2.4). This result is obtained by considering the integrated form of (2.3),

$$\dot{F}=zF'(z)G(z) , \tag{2.13}$$

where F', G, and F are given by (2.4), (2.6), and (2.11), respectively. Upon substitution of these functions and comparison of the logarithmic terms between the two sides of the equation one obtains (i) an EOM for the poles that coincides with (2.7), and (ii) an EOM for R_n

$$\frac{\dot{R}_n}{R_n}=-\frac{\dot{A}(t)}{A(t)} , \tag{2.14}$$

which confirms (2.12).

III. RESULTS FOR A CONSERVED NUMBER OF SINGULARITIES

We first present analytical results for the dynamics of the system when the number of singularities is conserved, i.e., when there is neither production nor annihilation of singularities, $N(t>0)=N(t=0)$. In the following we set the prefactor A in (2.4) to unity, which effectively corresponds to rescaling the growth area at each time step by a time dependent factor $A^{-2}(t)$. To gain insight into the dynamics of the many-body system we have integrated the trajectories of the singularities numerically. In each run we start from a random set of initial positions of the singularities. We confine the initial distribution of the poles closer to the origin than that of the zeros. This is done to prevent the strong short-range interaction between zeros and poles at the first few steps from leading to very high velocities already at the initial stages. So a pole is initially located randomly inside a circle of radius R_p, while the initial zeros are randomly located within a ring outside R_p. We have considered in each run between 5 and 20 pairs. But we have observed that the evolution of the system is not crucially dependent on this number. In general we find that the singularities "repel" each other as is demonstrated in Fig. 1, where both poles and zeros can be observed to move towards the unit circle. We find that these trajectories have *very small azimuthal velocities.*

Since the coupled EOM are strongly nonlinear and difficult to solve analytically we resort to calculations at particular limits. Inspecting the EOM we observe that when a zero (Z_n, say) comes sufficiently close to the unit circle the value of $Q_n\equiv Q(Z_n)$ varies proportionally to $1/(1-|Z_n|^2)$, while terms independent of Q_n in the EOM remain regular. The rapid variation of this quantity governs the EOM and therefore, to leading order in $(1-|Z_n|^2)$, we can solve the EOM by neglecting the terms that are not proportional to Q_n. Using this approximation we obtain the following EOM:

$$
\begin{aligned}
-d\ln Z_n/dt &\approx Q_n\left[\frac{1-3P_n/2Z_n}{1-P_n/Z_n}+\sum_{m'\neq n}\left[\frac{1}{Z_n-Z_{m'}}-\frac{1}{Z_n-P_{m'}}\right]\right], \\
-d\ln Z_{k\neq n}/dt &\approx \frac{Q_n}{Z_n}\left[\frac{1}{2}+\frac{1}{Z_k/Z_n-1}\right], \\
-d\ln P_k/dt &\approx \frac{Q_n}{Z_n}\left[\frac{1}{2}+\frac{1}{P_k/Z_n-1}\right].
\end{aligned} \tag{3.1}
$$

From the first of Eqs. (3.1) we can analyze the behavior of $Z_n=(1-\rho)e^{is_n}$, $\rho\ll 1$. Assuming all the other singularities are sufficiently farther away from the unit circle and that the distances between any two singularities is large compared to ρ we can solve these equations. We first note that if we rotate the system by s_n the singularities are all shifted by $e^{i(s-s_n)}$ and the values of Z_n and $1/Z_n$ become purely real. A careful analysis of the quantity Q_n/Z_n combined with the observation that the EOM should remain invariant under such rotation, gives that Q_n/Z_n should be independent of s_n. Namely, $Q_n = A_n e^{is_n}/(1-|Z_n|^2)$, where A_n is, in principle, a complex quantity independent of s_n. We therefore find

$$\rho(t) \approx \rho(t_0)\left[\frac{\tau-t}{\tau-t_0}\right]^{1/2}, \quad \tau \geq t \geq t_0 \tag{3.2}$$

where $\tau = \text{const}/\text{Re}\{A_n\}$, and

$$s_n(t) \approx s_n(t_0) + (\tau - t_0)\text{Im}\{A_n\}[\rho(t_0) - \rho(t)] \ . \tag{3.3}$$

We can see that $\rho \to 0$ after a finite time τ and the zero Z_n hits the unit circle, in agreement with the analysis of Shraiman and Bensimon [7]. The velocity of Z_n, $v_n \approx 1/\sqrt{\tau - t}$, *diverges* as $t \to \tau$. This divergence can be observed in the numerical results shown in Fig. 2. Moreover, the variation of the radial position according to (3.2) also agrees with the asymptotic behavior observed in Fig. 2, which was verified by plotting both the velocity and the radial position vs $(\tau - t)$ on a log-log plot. From (3.3) it may seem at first glance that the time dependence of s_n is similar to that of ρ, but a detailed analysis of Eqs. (3.1) confirms our observations that $s_n(\tau) = s_n(t_0)$ remains approximately constant along the trajectory. This observation can only agree with (3.3) if the prefactor of the time dependence in s_n, $\text{Im}\{A_n\}$, vanishes altogether. Thus we find from this particular analysis: (i) that the azimuthal angle of a zero approaching the unit circle hardly changes during the approach, and (ii) the quantity Q_n/Z_n tends to a *purely real* value in this limit. Our numerical calculations indeed support the latter result up to computational round off errors. We were not able to derive this result directly from the fundamental EOM although we suspect that such a first-principles calculation can probably be carried out.

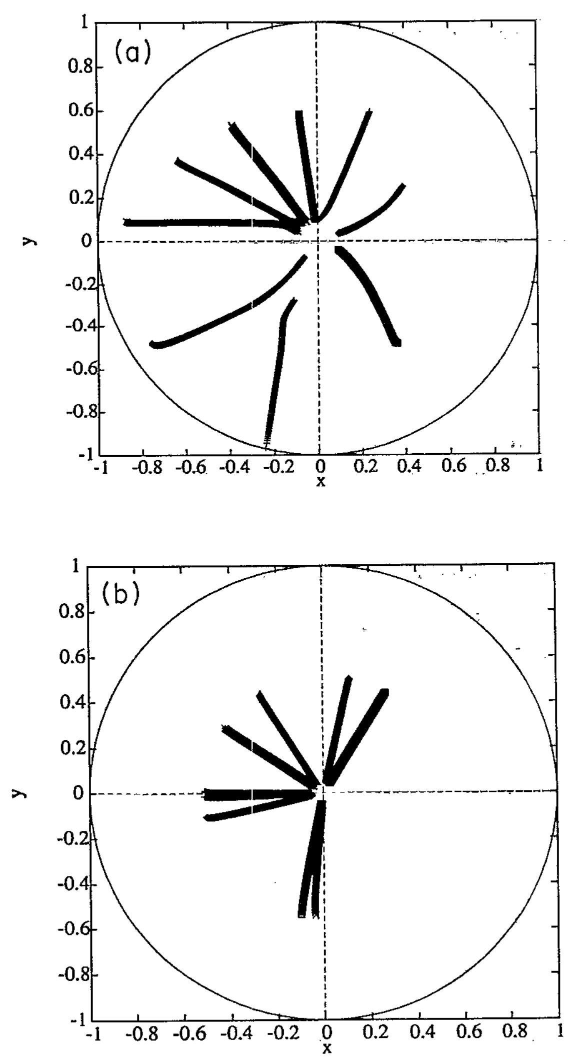

FIG. 1. Typical trajectories with time in the absence of surface effects: (a) zeros (b) poles.

Turning to the behavior of the other singularities in this limit we can immediately derive another result: it is straightforward to use the above calculations to find that for $k \neq n$

$$\frac{Z_k(t)}{Z_k(t_0)} = \frac{P_k(t)}{P_k(t_0)} \approx \frac{A_n}{2\rho(t_0)}(\tau - t_0)\exp\left[1 - \left[\frac{\tau - t}{\tau - t_0}\right]^{1/2}\right], \tag{3.4}$$

which shows that the rest of the singularities, both zeros and poles, are strongly affected by the proximity of Z_n to the unit circle. This strong effect can be probably better appreciated by noting that although the location of the singularities may be far from the unit circle, their velocities *diverge at exactly the same rate* as that of Z_n.

Next we turn to analyze the behavior of the curvature K as the zero Z_n approaches the unit circle. First, we want to calculate a general expression for K in terms of the locations of the singularities. This expression will be used later to discuss systems where the number of singularities is not conserved. The rate of change of the arclength along the unit circle, s, is related to the corresponding change in the length l along the physical boundary by the local value of $|F'|$. Denoting by ϕ the angle that the tangent to the real boundary makes with some reference axis, the local curvature is

$$K(s) = \frac{d\phi}{dl} = |F'|^{-1}_{z \to e^{is}} \frac{d\phi}{ds} \ . \tag{3.5}$$

Since ϕ is exactly the argument of the complex field E we obtain

$$\frac{d\phi}{ds} = \frac{d}{ds}\text{Im}\{\ln(zF')_{z \to e^{is}}\} \ . \tag{3.6}$$

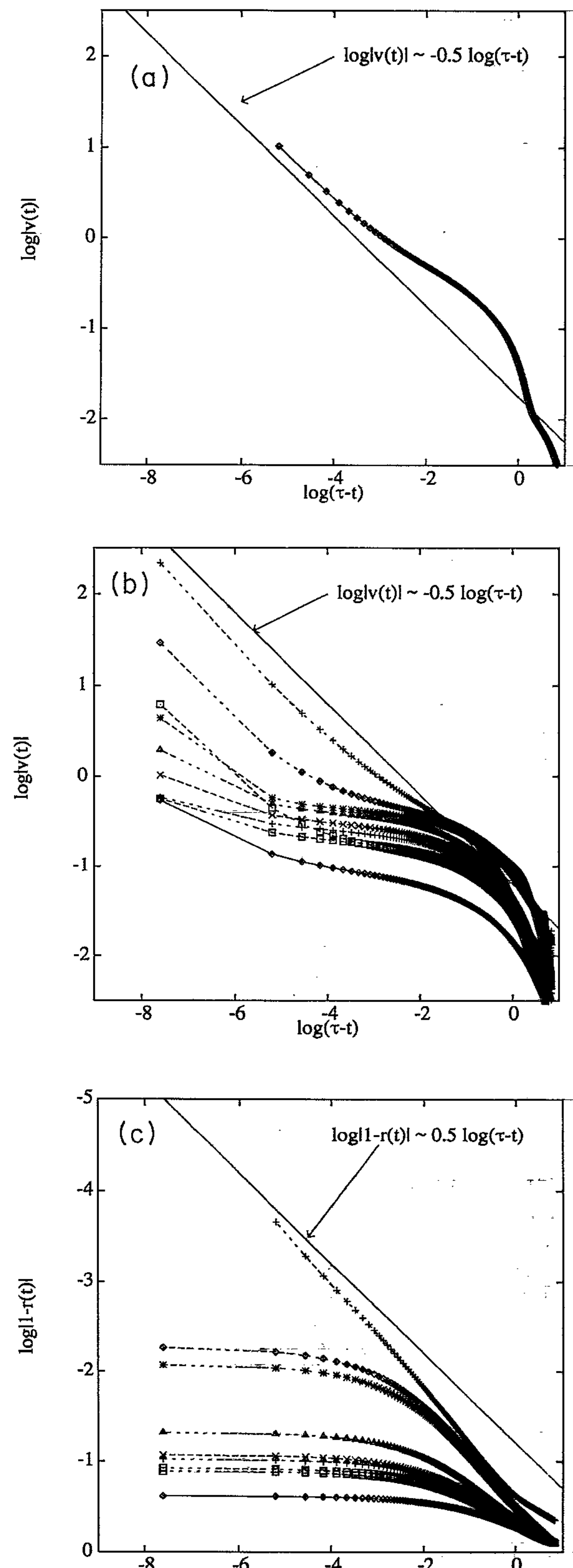

FIG. 2. In the absence of surface tension, natural logarithms of (a) the velocity of the zero closest to the unit circle; (b) the velocities of all the zeros; (c) the positions of all the zeros. Notice the $1/\sqrt{\tau - t}$ divergence of the velocity of the zero closest to the unit circle and the following behavior of the velocities of the rest of the zeros.

Combining Eqs. (3.5) and (3.6) and using some algebra, we find

$$K(s) = |F'|^{-1}\mathrm{Re}\left[1 + z\frac{d}{dz}\ln F'\right]_{z=e^{is}} = |F'|^{-1}\left\{1 + \mathrm{Re}\sum_m \left[\frac{1}{1 - Z_m e^{-is}} - \frac{1}{1 - P_m e^{-is}}\right]\right\} . \tag{3.7}$$

This expression is very useful because it allows for a clear and simple interpretation of the contribution of each of the singularities to the curvature. It is not surprising that the dominant contribution of a singularity whose argument is s_m within the curly braces is at the point along the surface that is parametrized by $s = s_m$. Expression (3.7) shows that when a pole $P_n = |P_n|\exp(is_n)$ is close to the unit circle it reduces the local curvature at the nearest point along the unit circle by approximately

$$\Delta K_n \approx - \prod_{\substack{m \\ P_m \neq P_n}} \left| \frac{e^{is_n} - P_m}{e^{is_n} - Z_m} \right| , \tag{3.8}$$

which is *regular* in the distance between the pole and the unit circle $1 - |P_m|$ as this distance goes to zero. On the other hand, when a zero $Z_n = |Z_n|\exp(is_n)$ approaches the unit circle its contribution to the local curvature *diverges* as

$$\Delta K_n \approx \frac{1}{(1 - |Z_n|)^2} \prod_{\substack{m \\ Z_m \neq Z_n}} \left| \frac{e^{is_n} - P_m}{e^{is_n} - Z_m} \right| . \tag{3.9}$$

This result can be combined with the above-calculated rate of approach of Z_n to the unit circle, when it is the most advanced zero, to yield the rate at which the curvature diverges at s_n

$$K(s_n) \approx \frac{\text{const}}{\tau - t} , \tag{3.10}$$

where $\tau > t$ is the time at which Z_n hits the unit circle. Thus the local curvature is *inversely proportional* to the time to "collision" of the zero with the unit circle. Differently put, the curvature increases as u_n^2, where u_n is the velocity of Z_n.

Next we write down the explicit form of the EOM for the local radius of curvature, $a(s) = 1/K(s)$, at any point along the boundary

$$\dot{a} = \sum_\Gamma \frac{\partial a}{\partial \Gamma}\dot{\Gamma} , \tag{3.11}$$

where Γ runs over all Z_n and P_n. We find that

$$\frac{\partial a}{\partial \Gamma} = \frac{\pm a}{2(z - \Gamma)} \times \left[1 + \frac{2z|F'|^{-1}}{(z - \Gamma)\{1 + i\tan[\arg(1 - \Gamma^*/z)]\}}\right] ,$$

where the + (−) sign corresponds to $\Gamma=P\,(Z)$. Upon substitution for $\dot\Gamma$ from the EOM (2.7) we obtain the required EOM for a in terms of the locations of the singularities

$$\frac{d}{dt}\ln a=\frac{1}{2}\sum_{\Gamma}\left[1+\frac{2z}{|F'|(z-\Gamma)}\right]\frac{\pm\dot\Gamma(\{\Gamma\})}{z-\Gamma}\;, \qquad (3.12)$$

with the appropriate sign as above. The study of this equation is of direct relevance to the statistics of the evolving surface, but is not the aim of this paper and we will address this direction elsewhere. Instead we now turn to introduce the tip-splitting mechanism and, using the above results, we analyze the modified many-body system.

IV. TIP SPLITTING

So far we have discussed the motion of singularities in the absence of surface effects up to the moment where the EOM no longer hold. As mentioned already, the breakdown of the formalism is associated with formation of a cusp singularity along the physical surface at a location that corresponds to the position s where a zero hits the unit circle. We now aim at modifying the dynamics so that the useful many-body description can be extended to arbitrarily long times. It is clear that the mechanism that prevents cusp formation in the real world (disregarding atomistic cutoffs for the purpose of the present discussion) is related to surface effects. Since the mathematical breakdown is always preceded by a local increase in the curvature at the incipient location of the cusp along the surface, we expect such effects to play a significant role in the modification of this formulation.

Surface tension originates from surface bending energy that the system has to expend as the curvature increases. Therefore, for real growths, where the energy is limited, a curvature-dependent term should be activated when the curvature becomes too large. In the mathematical plane the local increase in curvature corresponds to a singularity (generically a zero, as has been shown above) approaching the unit circle and therefore the activation of the surface effect should correspondingly be triggered by this approach. A related piece of information that is relevant to this issue is the following: it is frequently observed that the rate of tip splitting in growth phenomena depends on the local growth rate of the surface. The growth rate, in turn, is proportional to the gradient of the potential field via the constitutive relation (1.2). Since it is possible to relate between the field gradient and the curvature (through the fact that both $|F'|$ and K depend on the location of the singularities) it follows that *the splitting rate implicitly depends on the curvature.* Therefore we propose here to regard surface tension as effecting a *field* that interacts with the moving singularities. The mechanism that we suggest here works as follows: As a zero approaches the unit circle the corresponding location on the physical boundary develops an increasingly protruding arm. Numerous observations show that long arms cannot persist without eventually branching or tip splitting. We claim that tip splitting is the system's mechanism to *reduce the local curvature-dependent energy.* It is simple to prove this claim in the context of the present formulation. First we note that tip splitting in the physical plane corresponds in the mathematical plane to the reaction $Z\rightarrow 2Z+P$, namely, to a zero that is close to the unit circle (i.e., that is responsible to an already long arm and a high curvature term) spawning a new pair of a zero and a pole. The generated pole is located exactly between the two zeros. The newborn zeros correspond to the two newborn tips in the physical surface, while the pole effects a trough between these tips. This mechanism for singularities production is consistent with the constraining requirements on the map: (i) the number of poles equals the number of zeros at any time ("charge" neutrality); (ii) the sum rules (2.8) and (2.9) ("dipolar" neutrality) are obeyed at any time. For consistency, the new pole should be located at the position of the parent zero, while the two zeros, Z' and Z'' are located, immediately after the production event, at a predetermined small distance on either side of the pole. It is straightforward to calculate the change in the local curvature $K(s)$ immediately in front of the event. This calculation is carried out by expanding expression (3.7) in the small distance of the new zeros from their parent, and calculating the change in the curvature to the second order in this distance,

$$K(Z_n(e^{is}-\delta z),Z_{N+1}(e^{is}+\delta z),P_{N+1}(e^{is}),2N+2)$$
$$-K(Z_n(e^{is}),2N)=\delta K(\delta z)\;. \qquad (4.1)$$

We find that splitting in the radial direction $s'=s=s''$ increases $K(s)$, while a split in the azimuthal direction, $s'=s+\delta s$ and $s''=s-\delta s$ ($|Z|\delta s\ll 1$) *reduces* $K(s)$. Therefore, since surface energy increases with curvature, it should be favorable to produce singularities in the *azimuthal,* rather than in the radial direction. This process leads to a generation of two embryonic fingers advancing abreast immediately after the event. The stability of such competing fingers has been discussed recently in detail in the literature in the context of DLA [11] and we will not dwell on this analysis here. The above result strongly indicates that tip splitting is the system's way to reduce the local curvature term, and therefore the energetic cost of protruding arms.

Having offered an energetic explanation for tip splitting, we are left with the question of when and how is the spawning event triggered? The answer to this question cannot be universal and must depend on the particular system under study, namely, on the exact origin of the surface effects. One way to implement such a mechanism is to assume that the EOM of the singularities contain an *a priori* curvature term that is activated when the contribution of this singularity to the local curvature $\Delta K_n(s)$ increases above some threshold value K_c. As mentioned above, although the curvature at s_n depends on the location of all the singularities, it is nevertheless dominated by the closest zero Z_n [see Eq. (3.7)], and therefore, to a very good accuracy, we can use this threshold idea as a simplified rule, whose advantage is that it is *local.*

V. STATISTICAL ANALYSIS AND THE MULTIFRACTAL FUNCTION $f(\alpha)$

The above discussion focused on deterministically evolving surfaces and the randomness in the pattern could stem only from arbitrary choices of the initial locations of the singularities. In this section we discuss noise in such growth processes more generally and relate it to the probability of growth along the surface. Let us first point out what we expect from a statistical approach to this problem. In general, statistical analysis is prompted by phenomenological observations of a stable asymptotic distribution of some measurable quantity. For example, the thermodynamic properties of gases under given external conditions fluctuate narrowly around very well defined averages, indicating that whatever the route that led the system to its present state, the distributions of these properties do not change with time after some initial transient behavior. In other words, the system has reached a stable limit (asymptotic) distribution although microscopically the particles keep moving and interacting with each other.

Similarly, we seek a phenomenological stable distribution of a measurable quantity that characterizes the surface of the growth. One such candidate is the (normalized) distribution of the growth probability along the evolving surface. The local growth probability (or growth rate) in our problem is proportional to the local gradient of the field, $p(s)=|E(s)|/(2\pi Q)$, where $Q\equiv(1/2\pi)\oint|E(s')|ds'$ is the total charge along the interface in the context of the electrostatic problem. The distribution of this quantity has been found to be multifractal, namely, its moments scale with *independent* powers of the growth size [5]. A monofractal distribution corresponds to the case where the qth moment scales with a power that is linear in q. Such a distribution can be made independent of scale (viz., of stable limiting form) by normalizing the measure by its (scale-dependent) average. This simple method cannot convert a multifractal distribution into a scale-independent form, but such a conversion is possible [11,15,16]. The idea is to regard the logarithm of the measure (measure = the growth probability in the present context) divided by the logarithm of the scale as the new measure. Following this idea, we suggest the following candidate for a measurable quantity that flows to an asymptotic limiting form.

$$\alpha(s)=\frac{\ln p(s)}{\ln R_g}\ , \tag{5.1}$$

where R_g is the radius of gyration of the total physical growth. Note that the prefactor 2π normalizes Q conveniently to 1, as is shown below. We wish to study the asymptotic probability density of α, $P(\alpha)$. The crucial step is to relate $P(\alpha)$ to the spatial distribution of the singularities within the unit circle. This relation is found through expressing the magnitude of the electrostatic field at a point s $(0\le s<2\pi)$ in terms of the distribution of the singularities, $E(s)=|F'(z)|^{-1}_{z=e^{is}}$. The distribution of $|E(s)|$ is then given by

$$P(|E|)=\int\left\{\prod_{n=1}^{N}\mathcal{P}(\Gamma_n)d\Gamma_n\right\}\delta\{|E(z)|-|F'(z)|^{-1}\}\ , \tag{5.2}$$

where the locations of the singularities Γ have been introduced in Eq. (3.11) and where the limit $z\to\exp(is)$ is taken after integration. The distribution of α can be found from the distribution of $|E|$ as follows:

$$P(\alpha)=2\pi Q\ln R_g R_g^{\alpha}P(|E|)\ . \tag{5.3}$$

Thus all we need to do is find the probability density of $|E|$. It is well known that the knowledge of all the *odd* moments of the probability density of a measure on a finite support is sufficient to determine that probability density uniquely [17]. We therefore turn to calculate the moments $M_q\equiv\langle|E|^q\rangle$ next. This calculation then determines the multifractal spectrum through the asymptotic relation

$$f(\alpha)=\ln[P(\alpha)]/\ln R_g\ . \tag{5.4}$$

VI. THE DISTRIBUTION OF THE MOMENTS OF THE GROWTH PROBABILITY

In this section we first derive an exact expression for the qth moment of the growth probability for a given growth pattern. For a particular growth history this enables us to calculate the multifractal function $f(\alpha)$. Then, assuming that the asymptotic form of the distribution of the singularities is known, we discuss the ensemble average of the moments of the growth probability.

As mentioned already, for Laplacian growth, the qth of the growth probability density is [18]

$$M_q=\frac{1}{2\pi}\oint|E(l)|^q dl\ , \tag{6.1}$$

where the integration is carried out along the physical surface and where $Q=1$ has been used (see below). Transforming the integration to the mathematical plane by $dl=|F'(s)|ds$ and putting $E(l)=1/|F'(s)|$, we obtain

$$M_q=\frac{1}{2\pi}\oint|F'(s)|^{1-q}ds\ . \tag{6.2}$$

Substituting for $F'(s)$ from Eq. (2.4) and recalling that around the unit circle $s=-i\ln z$, we have

$$M_q=\frac{A(t)^{1-q}}{2\pi i}\oint\prod_{n=1}^{N}\left|\frac{z-Z_n}{z-P_n}\right|^{1-q}\frac{dz}{z}\ . \tag{6.3}$$

Using the fact the around the unit circle $z^*=1/z$, we end up with the following expression:

$$M_q=\frac{A(t)^{1-q}}{2\pi i}\oint\prod_{n=1}^{N}\left[\frac{n-Z_n}{z-P_n}\right]^{(1-q)/2}\times\prod_{n=1}^{N}\left[\frac{1-zZ_n^*}{1-zP_n^*}\right]^{(1-q)/2}\frac{dz}{z}\ . \tag{6.4}$$

The second product within the integrand is *analytic* inside the unit disk and therefore the integrand has only one simple pole at the origin and N singularities of order $(1-q)/2$ either at the locations of the zeros if $q>1$ or at the locations of the poles, if $q<1$. This indicates that the locations of the zeros dominate the positive moments, while the locations of the poles dominate the negative moments. This makes perfect sense—one does expect the protruding tips (characteristic to zeros close to the unit circle), that enjoy high growth rates, to dominate the high positive moments, while screened sites inside fjords (characteristic to poles close to the unit circle), which enjoy very low growth rates, are expected to dominate the high negative moments [11]. Moreover, it is not surprising that this change of roles occurs at $q=1$: From (6.3), it can be easily verified that $M_1=Q=1$, which is why this first moment serves as our normalization.

For simplicity, we consider here the odd moments, $q=2\nu+1$, where ν is a positive integer [19], and we first address the moments with $q>1$

$$M_q=\frac{A(t)^{-2\nu}}{2\pi i}\oint\prod_{n=1}^{N}\left[\frac{z-Z_n}{z-P_n}\right]^{-\nu}J(z)^{-\nu}\frac{dz}{z}\ . \qquad (6.5)$$

The quantity $J(z)\equiv\prod_{n=1}^{N}[(1-zZ_n^*)/(1-zP_n^*)]$ is analytic inside the unit circle and therefore the integrand contains a simple pole at the origin and poles of order ν located at the positions of the zeros. A straightforward calculation then gives

$$M_{q>1}=A(t)^{-2\nu}\left\{\left[\prod_{n=1}^{N}\frac{P_n}{Z_n}\right]^{\nu}+\frac{1}{\nu!}\sum_{n=1}^{N}\frac{d^{\nu-1}}{dz^{\nu-1}}\left[\frac{(z-P_n)^{\nu}}{zJ(z)^{\nu}}\prod_{k\neq n}\left[\frac{z-P_k}{z-Z_k}\right]^{\nu}\right]_{z=Z_n}\right\}. \qquad (6.6)$$

In particular, we calculate M_q for $\nu=1$, which corresponds to $q=3$ ($\nu=0$ is the normalization). This moment plays an important role because it has been found to yield the fractal dimension of the growth through $D_f=\ln M_3/\ln R_g$ [11,20]. In our formalism this moment can be exactly expressed in terms of the locations of the singularities

$$M_3=A(t)^{-2}\left[\prod_{m=1}^{N}\frac{P_n}{Z_n}+\sum_{n=1}^{N}\frac{Q_n}{2Z_n}\right]. \qquad (6.7)$$

For negative integer values of ν the integrand contains a simple pole at the origin, as before, but the poles of order $-\nu\equiv\bar{\nu}>0$ are now located at the positions of the *poles* of F'. Carrying out the integral (6.5) then yields

$$M_{q<1}=A(t)^{2\bar{\nu}}\left\{\left[\prod_{n=1}^{N}\frac{Z_n}{P_n}\right]^{\bar{\nu}}+\frac{1}{\bar{\nu}!}\sum_{n=1}^{N}\frac{d^{\bar{\nu}-1}}{dz^{\bar{\nu}-1}}\left[\frac{(z-Z_n)^{\bar{\nu}}J(z)^{\bar{\nu}}}{z}\prod_{k\neq n}\left[\frac{z-Z_k}{z-P_k}\right]^{\bar{\nu}}\right]_{z=P_n}\right\}. \qquad (6.8)$$

For example, for $\bar{\nu}=1$ ($q=-1$) one obtains

$$M_{-1}=A(t)^2\left[\prod_{n=1}^{N}\frac{Z_n}{P_n}+\sum_{n=1}^{N}\frac{\tilde{Q}_n}{P_n}\right], \qquad (6.9)$$

where

$$\tilde{Q}_n\equiv(P_n-Z_n)J(P_n)\prod_{k\neq n}\frac{P_n-Z_k}{P_n-P_k}\ .$$

Inspecting expressions (6.6) and (6.8) closely reveals that it is always the quantity $J(z)$ that causes singularities, which are close to the unit circle, to dominate the moments. For $q\geq 3$, $J(Z_n)$ contains a term of the form $(1-|Z_n|)^{q-2}$ in the denominator, which dominates these moments via the zeros that are very close to the unit circle ($1-|Z_n|\ll 1$). Similarly, for $q\leq -1$, $J(P_n)$ has the term $(1-|P_n|)^{-q}$ in the denominator, which make the *poles* that are nearest to the unit circle dominant. Below, we use these observations to analyze a limit in which the moments can be approximated by these dominant terms alone.

VII. THE DILUTE GAS APPROXIMATION

We have seen in Sec. III that if a zero Z_n, whose argument is s_n, is close to the unit circle it dominates the curvature at s_n, $K(s_n)$ [Eqs. (3.7)]. We have mentioned already that these zeros also dominate the positive moments of the growth rate distribution, and that poles that are close to the unit circle dominate the moments $M_{q<1}$. Generically, the latter are poles that were created in later generations because old poles are usually found in inner regions of the unit disk. Thus the interesting action takes place within some ring, $r_c<|z|<1$ (whose size may actually shrink with time—a point to be discussed elsewhere). To facilitate a simple calculation, we therefore neglect the effect of the singularities that are well inside the unit disk and consider only the effect of those that are no further from the unit circle than $1-r_c$. We assume that the distribution of the relevant singularities is dilute in the sense that the typical separation between neighboring singularities within this ring is larger than $1-r_c$. In this limit the qth odd moment ($q\geq 3$) is approximately

$$M_{2\nu+1\geq 3}\approx\frac{(2\nu)!A(t)^{2\nu}}{\nu!(\nu-1)!}\sum_n\mu_+^{\nu}\frac{1+|Z_n|}{|Z_n|^2}(1-|Z_n|)^{1-2\nu}+O(1)\ , \qquad (7.1)$$

where

$$\mu_+ \equiv \frac{Z_n^*}{(1+|Z_n|)^2} \frac{\prod_{k=1}^{N}(Z_n-P_k)(1-Z_nP_k^*)}{\prod_{k\neq n}(Z_n-Z_k)(1-Z_nZ_k^*)} ,$$

and where the sum runs over all the zeros whose $|Z_n| \geq r_c$.

A similar calculation for the negative values of ν gives

$$M_{1-2\tilde{\nu}\leq -1} \approx \frac{(2\tilde{\nu})! A(t)^{2\tilde{\nu}}}{\tilde{\nu}!(\tilde{\nu}-1)!} \sum_n \mu_-^{\tilde{\nu}} \frac{1+|P_n|}{|P_n|^2}(1-|P_n|)^{1-2\tilde{\nu}} + O(1) , \quad (7.2)$$

where

$$\mu_- \equiv \frac{P_n^*}{1+|P_n|)^2} \frac{\prod_{k=1}^{N}(P_n-Z_k)(1-P_nZ_k^*)}{\prod_{k\neq n}(P_n-P_k)(1-P_nP_k^*)}$$

has the same form as μ_+ when the poles are exchanged with zeros, and where the sum in (7.2) runs now over all the poles that satisfy $|P_n| \geq r_c$. It is intriguing to note that the negative and positive moments have exactly the same form when the zeros and the poles are interchanged. This observation suggests that differences between the behavior of the positive ($\nu \geq 1$) and negative ($\nu \leq -1$) moments can only arise from differences between the spatial distribution of zeros $f_z(\vec{Z})$ and that of the poles $f_p(\vec{P})$ within this outer ring.

We note that in both (7.1) and (7.2) the dominant part depends on the distance from the unit circle to a negative power $1-\nu$ (or $1-\tilde{\nu}$ for $\nu<1$) and we have explicitly separated this part from the more regular part in these expressions. Since these terms dominate the moments we consider the limit where the regular part is approximately constant. This allows us to rewrite (7.1) as

$$M_{2\nu+1\geq 3} \approx \frac{(2\nu)! A(t)^{2\nu}}{\nu!(\nu-1)!} \sum_n C_n^{\nu} D_n (1-|Z_n|)^{1-2\nu} , \quad (7.3)$$

where C_n and D_n depend on the location of Z_n only weakly. If the density of zeros is isotropic then the above reduces to the calculation of the *negative* moments of the density $f_z(\vec{Z}=xe^{is})$:

$$M_{2\nu+1\geq 3} \sim \int f_z(x) \frac{dx}{(1-x)^{2\nu-1}} \equiv \langle (1-|Z|)^{1-2\nu} \rangle . \quad (7.4)$$

Similarly, the calculation of the negative moments of the growth probability reduce, in this limit, to the calculation of the negative moments of $f_p(\vec{P}=xe^{is})$:

$$M_{1-2\tilde{\nu}\leq -1} \sim \int f_p(x) \frac{dx}{(1-x)^{2\tilde{\nu}-1}} \equiv \langle (1-|P|)^{1-2\tilde{\nu}} \rangle . \quad (7.5)$$

VIII. CONCLUSION

To conclude, we analyzed the problem of a one-dimensional interface evolving in a two-dimensional Laplacian field in terms of an equivalent many-body system. The particles of the many-body system are the singularities (poles and zeros) of the general conformal map that takes the interface into the unit circle at each instant of time. We discussed the constraints on the map and its constants of the motion. We wrote down the finite-dimensional set of first order ordinary differential equations that is equivalent to the infinite-dimensional partial differential equation that describes the motion of the interface and analyzed them numerically first. Calculating the trajectories of the singularities for various initial interfaces, we found that in the absence of surface tension the singularities "repel" strongly and that they "rush" towards the unit circle with an insignificant azimuthal velocity. Still without taking into account surface effects, we analyzed the behavior of singularities that are close to the unit circle. We found here that generically zeros hit the unit circle at a finite time, τ. The distance between the zero closest to the unit circle and unit circle $\rho(t)$ was found to decrease inversely proportional to the square root of the time to collision, $\rho(t) \sim \sqrt{\tau - t}$. The velocity of that zero then diverges as $1/\sqrt{\tau-t}$. Moreover, we found that the strong coupling between that zero and the other singularities causes the velocities of all the singularities to diverge during this fast approach at exactly the same rate.

We calculated the general equation of motion of the radius of curvature anywhere along the interface. This equation can be used as a basis for a statistical analysis of the curvature distribution along the interface, or in other words the morphology. The curvature immediately in front of an approaching zero was found to diverge at a rate proportional to $1/(\tau-t)$.

We next introduced a tip-splitting mechanism via production of new pole-zero pairs of singularities through the reaction $Z \rightarrow 2Z+P$. This production mechanism is triggered by proximity of a zero to the unit circle and hence a locally high curvature term. When the local curvature exceeds a threshold value, the zero that caused that high energy spawns a pole-zero pair at predetermined locations in the vicinity of the parent zero. The locations of the products of this reaction are arranged to reduce the high curvature, thus effectively reducing the local surface energy associated with high curvatures. We claim that tip splitting may therefore be the system's mechanism to "dissipate" locally high surface energies along the interface. As time goes on, the spawning of singularities takes place within an ever narrowing ring close to the unit circle. It is the distribution of the singularities within this ring that practically determines the morphology of the physical interface.

Thus we next addressed the relation between the general distribution of the singularities within the unit disk and the statistics of the interface. In particular, we analyzed the distribution of the growth probability $p \sim |E|$ along the interface in terms of the locations of the singularities. To obtain a stable asymptotic distribution, as seen in nature, we suggested monitoring the probability density of the quantity $\alpha = \ln p / \ln R_g$, $P(\alpha)$, which relates to the so-called multifractal function through $f(\alpha) = \ln P(\alpha)/\ln R_y$. We then calculated the exact moments of the growth probability distribution. By using the above relation these uniquely determine the multifractal spectrum. Our calculations show explicitly why the zeros dominate the positive moments, while the poles dominate the negative ones. To simplify the situation we considered the "dilute gas approximation," namely, when a narrow ring close to the unit circle contains singularities whose separation is typically larger than the width of the ring. Calculating the above moments in this regime, we found that the expressions for the positive and negative moments are identical if the locations of the zeros are interchanged with the locations of the poles. This indicates that the different behavior found in the literature between the positive and negative moments must originate from the different spatial distributions of the zeros and the poles inside the unit disk.

ACKNOWLEDGMENTS

R.B. acknowledges partial support of the Office of Basic Energy Sciences, U.S. Department of Energy under Grant No. DE-FG02-92ER14275 during his stay at Princeton University. This work started at Cavendish Laboratory, U.K., continued while R.B. was at the Princeton Materials Institute, and was concluded when R.B. was at Los Alamos National Laboratory.

[1] T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981); for a review see, e.g., P. Meakin, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, New York, 1988), Vol. 12; T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).

[2] L. Pietronero, A. Erzan, and C. Evertsz, Phys. Rev. Lett. **61**, 861 (1988); Physica A **151**, 207 (1988).

[3] T. C. Halsey and M. Leibig, Phys. Rev. A **46**, 7793 (1992); T. C. Halsey, Phys. Rev. Lett. **72**, 1228 (1994).

[4] L. A. Turkevich and H. Scher, Phys. Rev. Lett. **55**, 1026 (1985); R. C. Ball, R. M. Brady, G. Rossi, and B. R. Thompson, *ibid.* **55**, 1406 (1985); R. C. Ball, Physica A **104**, 62 (1986).

[5] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974); T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986); T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[6] L. A. Galin, Dokl. Akad. Nauk SSSR **47**, 246 (1945); P. Ya. Polubarinova-Kochina, *ibid.* **47**, 254 (1945); Prikl. Mat. Mekh. **9**, 79 (1945).

[7] B. Shraiman and D. Bensimon, Phys. Rev. A **30**, 2840 (1984).

[8] L. Paterson, J. Fluid Mech. **113**, 513 (1981); L. Paterson, Phys. Fluids **28**, 26 (1985); D. Bensimon and P. Pelce, Phys. Rev. A **33**, 4477 (1986); S. Sarkar and M. Jensen, *ibid.* **35**, 1877 (1987); B. Derrida and V. Hakim, *ibid.* **45**, 8759 (1992).

[9] S. D. Howison, J. Fluid Mech. **167**, 439 (1986).

[10] But note that this stability may be violated at the tails of the distribution as discussed in R. Blumenfeld and A. Aharony, Phys. Rev. Lett. **62**, 2977 (1989); A. B. Harris, Phys. Rev. B **39**, 7292 (1989); A. B. Harris and M. Cohen, Phys. Rev. A **41**, 971 (1990); J. Lee and H. E. Stanley, Phys. Rev. Lett. **61**, 2945 (1988); H. E. Stanley, A. Bunde, S. Havlin, J. Lee, E. Roman, and S. Schwarzer, Physica A **168**, 23 (1990); S. Schwarzer, J. Lee, S. Havlin, and P. Meakin, Phys. Rev. A **43**, 1134 (1991).

[11] R. C. Ball and R. Blumenfeld, Phys. Rev. A **44**, R828 (1991).

[12] R. Blumenfeld, Phys. Rev. E **50**, 2952 (1994).

[13] N. Robidoux and R. Blumenfeld (unpublished).

[14] S. Richardson, J. Fluid Mech. **56**, 609 (1972).

[15] B. Fourcade and A.-M. S. Tremblay, Phys. Rev. **36**, 2352 (1987).

[16] A. Aharony, R. Blumenfeld, and A. B. Harris, Phys. Rev. B **47**, 5756 (1993); J. Adler, A. Aharony, R. Blumenfeld, A. B. Harris, and Y. Meir, *ibid.* **47**, 5770 (1993).

[17] B. Fourcade and A.-M.S. Tremblay, Phys. Rev. B **36**, 8925 (1987); B. Fourcade and A.-M. S. Tremblay, Phys. Rev. A **36**, 2352 (1987); A. Aharony, R. Blumenfeld, P. Breton, B. Fourcade, A. B. Harris, Y. Meir, and A.-M. S. Tremblay, Phys. Rev. B **40**, 7318 (1989).

[18] It should be pointed out that this treatment can be readily generalized to dielectric-breakdown models, where the growth probability is proportional to $|E(l)|^{\eta}$.

[19] This choice simplifies the calculation of the resulting integrals because they involve only summation over residues. A more general calculation calculation involves branch cuts between pairs of singularities, which can be calculated exactly, but will not contribute to the understanding of the results. The generalization of the calculation in the text to an arbitrary real value of ν will be addressed elsewhere.

[20] T. C. Halsey, Phys. Rev. Lett. **59**, 2067 (1987).