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Onset of scale-invariant pattern in growth processes: the cracking problem

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We discuss a possible mechanism for the onset of scale-invariant pattern when a cracking structure propagates in a continuous medium. We show that sufficiently close to the tip of an evolving arm the stress field is insensitive to the shear on the boundaries far away. We predict the number of major arms given the local relation between the growth rate and the stress field. We find oscillatory modes periodic in the logarithm of the distance from the tip of the pattern. We argue that these solutions lead to initiation of log-periodic corrugations which are unstable and hence develop into fully grown sidebranches with the same spacing pattern. This pattern is scale-invariant and hence this analysis provides a mechanism for the onset of self-similarity in these structures, a phenomenon observed in many simulated and real systems. The relation to the pattern formed by a diffusion controlled growth is discussed.

1. Introduction

The field of pattern formation has enjoyed a burst of activity in recent years from the point of view of statistical physics. In particular, structures displaying scale-invariance on many length scales have been investigated intensively. Two particular examples in two dimensions (2D) are diffusion controlled growth and quasi-stable cracking formation. The former is governed by a harmonic field, while the latter by the biharmonic equation. One can imagine a structure grown stochastically by a general field, where the growth is governed by some local relation between the rate of advance of the front and the magnitude of the field. So progress in any of these problems may give insight into dealing with other growth processes. In spite of the large body of work done on such problems, there is little understanding why the asymptotic structures evolve into the (fractal) patterns observed in so many real and simulated stochastic growths. In other words, there is no consistent analysis that starts from first principles and leads to formation of self-similar patterns.

Recently, in trying to understand some computer simulations [1,2], we have addressed this issue directly in the context of the propagation of a quasi-stable cracking pattern [3]. The problem of the mechanics and dynamics of propagating cracks is of theoretical significance, as well as enjoying a large variety of possible applications and technological uses. A crack initiated in a brittle material propagates fast due to local breakdown at the running tip, while in ductile media such propagation may be very slow. It follows that the final pattern depends significantly on the ratio of the typical time of propagation (to be defined more precisely below) to the relation time of the stress field around the crack. The work described here focuses on the quasi-stable (adiabatic) regime when the crack grows slowly, allowing the elastic stress field to relax so that it can be considered at equilibrium at any time. The above computer simulations of this process usually assume an underlying lattice structure in which a crack is initiated, and whose rate of growth at its front, v, is a function of the local stress $f(\sigma)$. These simulations have shown that the resultant morphology in two dimensions (2D) is fractal with a fractal dimension that depends much more on the model assumed rather than on the boundary conditions (BC). We focus on quasi-stable cracking in an isotropic continuous medium. This should provide insight into the mechanics of propagating cracks in general and to the understanding of the aforementioned simulations in particular. Our results are the following: (i) Modelling the envelope of the growing pattern by a wedge, we first identify the physical origin of the dominant two singularities at the tip of this wedge. The main singularity is of extensional character; provided there is anisotropy in the crack structure to couple it, this dominates even when the externally applied stress is shear. (ii) We use the main singularity to estimate the number of major arms of the growth. (iii) We find that within a substantial range of wedge angles all higher order corrections to the stress field at the edges of the envelope oscillate with periodicity in log of the distance from the tip. These oscillations are argued to initiate corrugations on the envelope of the pattern with the same periodicity. (iv) We show that these corrugations are unstable and grow outwards to become fully grown sidebranches with spacings selected by the static solutions. This may provide a mechanism for the onset of scale-invariant pattern in these systems. We discuss the difference and the resemblance between the biharmonic and the harmonic cases.

2. The stress field near the envelope of the pattern

We consider a cracking pattern forming in a homogeneous and isotropic elastic medium. Contrary to other approaches [4], we start from an already fully developed pattern and analyse its further evolution. The cracking process generally forms a very complicated and ramified structure, whose interior carries very little stress. Therefore, to a good approximation, we can envisage the structure as enveloped by an imaginary surface (line in 2D) and consider this envelope as an effective boundary for the stress field outside the structure. Following others (and ocular evidence in 2D), we approximate this envelope as a wedge of head angle $2\beta=2\pi-2\alpha$ (see inset in fig. 1). One can choose other shapes, but although highly idealised, this shape is believed to capture the essential features of crack patterns in many cases, as well as being con-

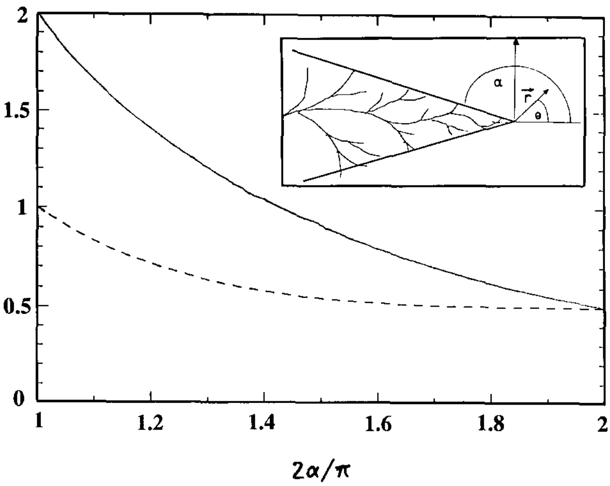

Fig. 1. The smallest $m(\alpha)$ solution for the odd (m_- in full line) and even (m_+ in dashed line) version of eq. (4). Note that $m_+ < m_-$ for all α. Inset: the wedge-shaped envelope of the crack.

venient for analytical calculations. It is convenient to represent the stress tensor in terms of the Airy stress function (ASF) Φ. The stress tensor is symmetric and is derived from Φ via $\sigma_{xx} = \partial_{yy}\Phi$, $\sigma_{yy} = \partial_{xx}\Phi$ and $\sigma_{xy} = -\partial_{yx}\Phi$. Since the interior of the cracking pattern carries very little stress, we hereafter neglect it altogether and use free boundary conditions on the edges of the wedge envelope $\partial_{x'x'}\Phi = \partial_{x'y'}\Phi = 0$, where x' and y' are, respectively, the normal and tangent directions to the surface at any point along it. These can be rewritten in the form

$$(\boldsymbol{x}' \cdot \nabla)\nabla\Phi = \boldsymbol{0}, \tag{1}$$

leading to $\nabla\Phi$ being a constant vector along the edges. This constant corresponds to a rigid displacement of the entire system, which is of no interest to us here and we choose $\nabla\Phi = \boldsymbol{0}$ along the surface. This gives Φ = constant along the surface, which is again of no significance to us, and is therefore chosen here to vanish as well. The other boundaries of the system are at $y = \pm\infty$ with the BC there being an arbitrary combination of extension and shear. In an isotropic medium, Φ satisfies the biharmonic equation

$$\nabla^4\Phi = 0. \tag{2}$$

Eqs. (1) and (2) determine uniquely the stress field in the exterior of the pattern. Both these equations are independent of the compressibility of the material and hence of its specific properties, thus demonstrating directly the universality of the solutions to Φ. This combined with the simplifications in the following calculations are the reasons for using the ASF rather than the more commonly used stream function.

The ASF can be separated into an even (Φ_e) and an odd (Φ_o) contribution, which can be expanded near the stress-free wedge envelope as

$$\Phi_e(r,\theta)=\sum_m \{p_m \cos[(m-1)\theta]+q_m \cos[(m+1)\theta]\}r^{m+1}\ , \qquad (3)$$

where r and θ are, respectively, the distance from the tip of the wedge and the azimuthal angle (see inset in fig. 1). The corresponding equation for the odd contribution is similar to (3) with the "cos" terms replaced by "sin". Inspecting (3) it is important to note that it admits solutions where m is *complex*, $m=\mu+i\nu$. The even (odd) part of Φ corresponds to a pure extension (shear) applied on the boundaries at $y=\pm\infty$. The complete solution consists of a linear combination of both parts with the above BC. Applying the BC we can eliminate the coefficients p_m/q_m and obtain two equations for m in the form [2,5]

$$\frac{\sin(2m\alpha)}{m}\pm\sin(2\alpha)=0\ , \qquad (4)$$

where the + (−) sign corresponds to the even (odd) contribution.

Near the tip of the wedge, the smallest value of μ governs the behaviour. Nevertheless, higher order terms, which act only as corrections near the tip, are shown below to play a significant role in the evolution of the growth. Hence we study the general solutions in some detail. Two straightforward solutions are $m=0$ and 1, which correspond to a constant added, respectively, to the displacement and to the stress fields. These trivial solutions are of no interest to us here. We first find that the smallest solution for both $\Phi_e(m_+)$ and $\Phi_o(m_-)$ are purely real for *all* α. For $2\alpha<2\alpha^*\approx 1.43\pi$, m_- increases from 1 to 2 as $2\alpha\to\pi$, indicating a vanishing contribution near the tip and suggesting that the shear on the boundaries far away is irrelevant. For $2\alpha>2\alpha^*$ the odd solution drops below 1 and tends to $\frac{1}{2}$ when $2\alpha\to 2\pi$, thus diverging as $r\to 0$. The smallest even solution to (4), m_+, is smaller than one all over the range $\pi<2\alpha<2\pi$, and hence diverges at the tip for any α. We also find that $m_+<m_-$ for all $\alpha<\pi$, indicating that the even solution always dominates near the tip. In the conventional model of a line crack, corresponding to $\alpha=\pi$, $m_+=m_-=\frac{1}{2}$, which obscures this important observation. It follows that sufficiently close to the tip the field is governed by the extension tension at the boundaries far away, while the shear there only modifies the singularity slightly. This result can be interpreted as indicating that within the interior of the structure, the detailed nature of the BC far away is locally obscured due to the convoluted pattern and hence the universal solution is insensitive to these BC. Following this observation we can deduce that in sufficiently big systems the morphology of the growth depends very weakly on the nature of the external BC. In particular, this means that the fractal dimension found in recent computer simulations [1,2] should not depend on those BC. Indeed, the difference in the fractal dimension due to differing BC was found to be very small, which supports this conclusion and may imply that this difference could be attributed to finite size effects. The lowest two singularities thus describe a renormalisable scaling behaviour and their dependence on α is shown in fig. 1.

Turning to higher order corrections, we find that when $\sim 1.155\pi < 2\alpha < \sim 1.75\pi$, m has an *imaginary* component for *all* higher order terms. This gives rise to an oscillatory term $r^{\mu} \exp(i\nu \ln r)$. For 2α outside this range more real solutions to m appear in lowest order terms (see fig. 2 and its caption), until when $2\alpha \to \pi$, 2π the roots m_i tend to the set of all real integers and half-integers and the oscillatory modes disappear altogether. Thus the oscillations are pronounced within the above range of angles and when the edge is either too sharp, or too blunt, these modes are suppressed.

3. Stability analysis and branching

Due to the universal nature of the solutions for the stress field, the above discussion applies to any elastic continuum system. Nonuniversality sets in via the local relation between the growth rate of the front v and the stress perpendicular to this front, $f(\sigma_\perp)$. Since one usually expects $\partial v / \partial \sigma_\perp > 0$ then initiation of new cracking along the envelope of the crack is favoured at locations where the stress is locally peaked. This implies that the above oscillatory modes initiate cracking that follows the same periodicity in ln r as the stress field.

Next we need to establish that once initiated, the embrionic cracks survive to eventually contribute to the global pattern. This we do by analysing the stability of a cor-

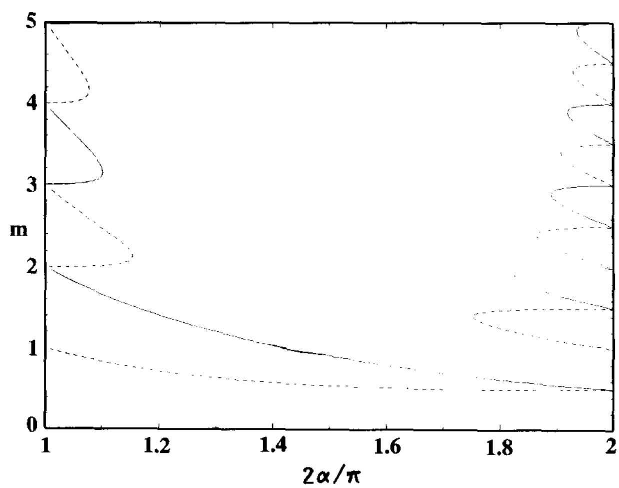

Fig. 2. The number and values of the real solutions to the odd (even) version of eq. (4) can be found at any α by the intersections of a vertical line at the correct value of $2\alpha/\pi$ with the shown full (dashed) lines. In the range $1.155 < 2\alpha/\pi < 1.75$ only the lowest two are real (disregarding the trivial solution $m = 1$), and all higher orders are complex.

rugation on the envelope of the cracking pattern and show that it propagates faster than the advance of its immediate neighbourhood. For simplicity we consider the stability of a stress-free flat surface, aligned along the x-axis with a tensile stress applied to a boundary at $y=\infty$. This analysis is expected to hold for any reasonably smooth surface when corrugated perturbatively. The perturbed surface can be represented by

$$y(x) = \sum_{k=-\infty}^{\infty} Y_k \exp(ikx) , \tag{5}$$

with $Y_0 = 0$ at time $t = 0$, and the coefficients $Y_k \ll 1$. To first order in Y_k, we find

$$\Phi = \sigma_0 y(x) \left(\tfrac{1}{2} y - \sum_k Y_k \exp(ikx - |k| y) \right), \tag{6}$$

where σ_0 is the stress perpendicular to the unperturbed surface. This leads to

$$\sigma_\perp = \partial_{yy} \Phi = \sigma_0 + 2\sigma_0 \sum_k |k| Y_k \exp(ikx) . \tag{7}$$

Considering now the local relation $v(\boldsymbol{r}) = f(\sigma_\perp(\boldsymbol{r}))$, expanding in Y_k and keeping only the lowest term, we find

$$v = \partial_t y(x) = f(\sigma_0) + 2\sigma_0 \, \partial_\sigma f(\sigma_0) \sum_k |k| Y_k \exp(ikx) . \tag{8}$$

Using (5) and (8) and equating term by term we get an equation for Y_k that can be solved to yield

$$Y_0(t) = f(\sigma_0)\, t , \qquad Y_{k \neq 0}(t) = Y_k(0) \exp(t/\tau) , \quad \tau = 1/[2|k| \sigma_0 \, \partial_\sigma f(\sigma_0)] . \tag{9}$$

As mentioned above, $\partial_\sigma f(\sigma_0) > 0$ and hence Y_k increases exponentially in time, making any k-wave corrugation unstable. The sharper the corrugation, the larger k and the more unstable is the k-wave. The particular form used in the numerical simulations, $f(\sigma) \sim |\sigma|^\eta$ [1,2], yields $1/\tau \sim 2\eta |k| \sigma_0^\eta$.

Since the value of τ defines a time scale relating to the crack's propagation, we can now exactify the condition of quasi-stability (also defining the range of applicability of our analysis) by requiring the stress field to equilibrate faster than τ. From (9) we see that τ decreases with $|k|$ indefinitely, so for the growth to remain quasi-stable another mechanism must exist (e.g., surface energy) that prohibits very sharp corrugations. If the cutoff is $\lambda_c = 2\pi/k_c$, corresponding to τ_c, then the stress field relaxes faster than the propagation of the pattern when the sound velocity in the medium is larger than $\lambda_c/\tau_c \sim 4\pi\sigma_0 \, \partial_\sigma f(\sigma_0)$. Evidently this condition is hardest to satisfy near the tips where the stress is largest.

Now consider the effect of screening competition between neighbouring major arms on the resultant pattern [#1]. We imagine a growth with $2N$ equivalent major branches and analyse the pattern's stability towards small modulation of one of them from length R to $R+\delta R\cos(N\theta)$. Starting from the ASF for the unmodulated growth:

$$\Phi(r,\theta)=R\sum_{n=1}^{\infty}[a_n+(r/R)^2 b_n](r/R)^{-n}\cos(n\theta)+R[a_0+b_0(r/R)^2]\ln(r/R). \quad (10)$$

Disregarding the logarithmic term, which cannot affect the angular modulation, the ASF for the modulated growth can be written as $\Phi'(r,\theta)=\Phi(r,\theta)+\delta\Phi(r,\theta)$, where

$$\delta\Phi(r,\theta)=\delta R\sum_n [na_n+(n-2)b_n(r/R)^2](r/R)^{-n-N}\cos[(N+n)\theta]$$

satisfies the biharmonic equation and near the tips ($r=R$ and $\cos(N\theta)=\pm 1$) equals $\pm\delta R\,\partial\Phi/\partial r$, thus advancing or retarding the ASF solution by δR radially. We now compare the behaviour of the new ASF at a cutoff radius $\rho \ll R$ from the tips of the modulated growth with the old ASF at a distance ρ from the unmodulated tips and we find (to first order in $\delta R/R$)

$$\Phi'(R+\delta R\cos(N\theta)+\rho,\theta) = \Phi(R+\rho,\theta)\left(1+(N-1)\frac{\delta R}{R}\cos(N\theta)\frac{\partial\ln\Phi(R+\rho,\theta)}{\partial\ln\rho}\right). \quad (11)$$

Using the stress singularity near the tip $\Phi(R+\rho,\theta)\sim\rho^{1+m_+}$, the derivative on the rhs of (11) becomes $m_+ + 1$. When the structure grows from R to $R'=R+\delta R$ its rate of growth changes from v to $v'=v+\delta v$ and it is easy to see that $\delta R'/R'=(\delta R/R)(1+\delta v/v-\delta R/R)$. So the growth is unstable when $\delta v/v>\delta R/R$ and vice versa. But self-similarity implies that at each length scale the relative growth $\delta R/R$ rate cannot change giving $\delta v/v=\delta R/R$ (marginal stability). For example, when $v\sim|\sigma|^{\eta}$, $\delta R/R=\eta$. Substituting all this into (11) yields that $\delta R/R$ increases with growth when $\eta(N-1)(1+m_+)>1$ and hence the highest sustainable modulation corresponds to the maximal stable number of arms, giving

$$N^*=1+1/[\eta(1+m_+)]. \quad (12)$$

Since $\frac{1}{2}\leqslant m_+\leqslant 1$ we can bound N^* by $2+1/\eta\leqslant 2N^*\leqslant 2+4/3\eta$. For $\eta=1$ this puts $2N^*$ very close to 3. The quantity π/N^* should be interpreted as the smallest stable angle between the direction of growth of neighbouring major arms.

4. Discussion and conclusions

Scale invariance (i.e., invariance under the transformation $r\to r'=\lambda r$) amounts to

[#1] For the harmonic field parallel, see e.g. ref. [6].

translational invariance in ln r space ($\ln r \to \ln r' = \ln \lambda + \ln r$). So periodicity in ln r is the fingerprint of scale-invariant structures. In the context of aggregation and the formation of cracking patterns this corresponds to geometrically spaced sidebranches, which are usually observed in real systems. Hence, the above oscillatory solutions must bias the growth strongly towards such a pattern as follows: Following the local maxima in the stress field, initial cracks appear as corrugations along the envelope of the pattern with ln r periodicity which traces that of the stress field. Being extremely unstable, as we have shown, these corrugations develop into fully grown sidebranches, freezing the original spacing pattern.

It is interesting to note that the periodic change of sign of the perpendicular stress along the edge, combined with relation (9), indicates alternating regions of relative stability (suppressed growth) and instability (enhanced growth). This may provide a mechanism for suppressing growth locally, in addition to the (currently believed to be unique) termination of cracking by screening. It should be emphasised that although the oscillatory modes only modify the leading power-law behaviour in the static solutions to the stress field, their contribution to the dynamic evolution of the pattern may be considerable. This is because the dominant singularities cannot affect any periodicity in the field along the edges, while the oscillating modes can and consequently initiate local cracking, which are strongly amplified by the field. Thus it is possible that these very terms are actually more important in determining the resultant pattern.

Let us discuss the relation of the above analysis to diffusion limited aggregation (DLA). In DLA the growth is governed by the harmonic (rather than the biharmonic) field, and so cannot accommodate static solutions that oscillate in ln r. Nevertheless, both processes result in a fractal pattern caused by continuous tip splitting. This implies that the origin of the fractal pattern of major sidebranches need not originate exclusively from static solutions, and there can also exist another selection mechanism that influences the growth process. Such an independent mechanism would compete, in the case of the cracking problem, with the periodicity dictated by the static solutions. This competition need not be destructive, but rather such a mechanism can act only as a filter, resonating with a certain mode and selecting a particular value of ν. The selection mechanism originates from the screening competition between branches on the strength of the field (which determines the influx of particles in the case of DLA, and the growth rate in the cracking propagation) and can be described in terms of geometrical constraints on the final structure [6]. Such a mechanism may in principle be independent of the field and hence apply (at least qualitatively) to growth in a general field. Hence its study is an important step towards understanding patterns formed by stochastic growth in general. It is plausible that the mode selected in the biharmonic case will correspond to one of the low order terms in expression (3), but it need not necessarily be the lowest, as a selection mechanism resonating with one of the higher order terms may amplify it to dominate the pattern.

In the absence of static solutions in DLA oscillating in ln r, as above, the external selection mechanism remains the sole factor governing the growth. Therefore one should study this mechanism in the context of DLA [7] and then apply the insight gained from it to the cracking problem.

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