Fluid flow in a random porous medium: A network model and effective medium approximation

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(Received 16 February 1987; accepted for publication 27 April 1987)

Fluid flow through a random porous medium is discussed in the context of a network model of a diluted array of planar cracks by an effective medium approximation. We find the threshold concentration of cracks p_c above which flow occurs. It turns out to be much higher than the bond percolation threshold. The existing cracks are assumed to have a range of thicknesses. The flow permeability is found as a function of the concentration for a number of crack thickness distributions. Near the threshold, anomalous critical behavior, in the form of a nonuniversal critical exponent for the permeability, is found to occur even for a family of nonsingular thickness distributions.

I. INTRODUCTION

The transport properties of fluid flow in a random porous medium are difficult to describe by the usual homogenization techniques that assume the existence of a typical volume large enough for local averages but small compared with the system size. Such a system is our model of fractured rock: a simple cubic lattice in which each square facet, or plaquette, represents a quasiplanar crack of thickness t or zero with probability p or $1-p$, respectively. In reality, there are also thin filamentary cracks, but those are quasi one dimensional, and it seems plausible that their contribution to the flow is negligible relative to that of the planar cracks.

The problem of flow through a fractured medium has been discussed recently using a number of different models.[1–4] In all of these models the permeability k vanishes below some threshold crack concentration $p=p_c$, and increases monotonically with p above it.

The aim of this paper is to calculate the permeability of percolating planar cracks in an effective medium approximation.[5,6] The model we apply is of square plaquettes in a simple cubic lattice. The array of plaquettes is diluted, i.e., only a fraction p of the plaquetttes permits flow while the others are blocked (or do not exist). We find that p_c is higher than the usual bond percolation threshold. We derive the explicit variation of the permeability with p for the case of a simple binary distribution of thicknesses (i.e., all existing plaquettes have a fixed thickness t) and also for some continuous distributions of the thickness [i.e., the thickness of an existing plaquette is chosen according to a given probability density function (PDF)]. It is well known that effective medium approximations yield incorrect results for critical exponents near the percolation threshold. Nevertheless, they indicate qualitatively the effects that various PDF's will have on the critical behavior. Furthermore, away from p_c such approximations often become quantitative and can thus be very useful in the context of flow in a porous medium over a wide range of conditions.

This article is organized as follows. In Sec. II we derive the formal equation for the macroscopic effective permeability by applying effective medium theory (EMT) to our model. In Sec. III we consider several PDF's with which the thickness of a crack may be distributed. These are the binary PDF,

$$P(t)=(1-p)\delta(t)+p\delta(t-\tilde{t}),$$

the constant PDF,

$$P(t)=(1-p)\delta(t)+p(\text{const}),$$

and some singular power law (SPL) distributions that attribute increasingly larger weights to the smaller thicknesses.[7–9] In Sec. IV we discuss the results.

II. A MODEL OF CRACK PERCOLATION AND ITS SOLUTION BY EMT

Consider a simple cubic lattice with nearest-neighbor lattice spacing w. Microscopic cracks in the rock are represented by the square faces of each unit cube, called plaquettes (see Fig. 1). Each plaquette may either be present with probability p or absent with probability $1-p$. An existing plaquette may be thought of as a square $w\times w$ crack of thickness $t\ll w$. When a pair of existing plaquettes have a common edge, fluid may flow from one to the other. Contact only at a vertex is not enough for transmission of fluid and such pairs are considered to be unconnected (see Fig. 1). In order that fluid may flow through the lattice, the microcracks must form a percolating cluster, i.e., a cluster that has continuous paths between opposite edges of the lattice. Thus, we are led to consider the percolation of plaquettes in a simple cubic lattice. In previous treatments of random net-

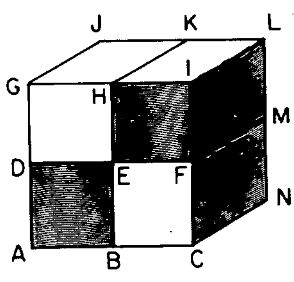

FIG. 1. A section of a simple cubic lattice showing present plaquettes (ABED, CNMF, FMLI, and EFIH) and absent ones (BCFE, DEHG, GHKJ, and HILK). FMLI is connected to both CNMF and EFIH, while EFIH and ABED are unconnected.

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works that usually focused on the properties of electrical conductivity or elastic stiffness, only the percolation of sites or bonds was considered. However, for the problem we are treating, plaquette percolation seems to be a more appropriate model, as pointed out in Ref. 3. We treat this model by applying an effective medium approximation. In this approximation, each plaquette is considered to be embedded in a uniform lattice of effective plaquettes, all of which have the same thickness t_e, or flow permeability k_e, determined by a self-consistency argument. A single plaquette is replaced by one of thickness t (with permeability k) which alters the uniform flow pattern in the immediate neighborhood of that plaquette. The properties of the effective plaquette are determined by requiring that the flow pattern remains unchanged on the average.

We implement this idea by modifying the EMT of the bond network as presented by Kirkpatrick.[10] Assume a large diluted cubic lattice of plaquettes. Apply a pressure head between its two opposite boundaries. The resulting distribution of pressure is considered to be a superposition of a uniform "external field" and a local fluctuating field having a zero average. The average effects of this lattice are represented by the aforementioned effective lattice with plaquettes of identical thickness t_e. Consider a plaquette of the uniform effective lattice which lies in the direction of the uniform external field and change its thickness to t. Consequently its permeability changes to k, distorting the flow field in that plaquette and also in its neighborhood. In order to compensate for these changes, we introduce an additional external current i_0 which flows in and out at the two edges of the plaquette t, which are perpendicular to the original uniform flow field (see Fig. 2). If we denote by P_e the uniform pressure drop across each effective plaquette lying along the field, then i_0 is given by

$$i_0 = P_e (k_e - k). \tag{1}$$

The extra pressure drop P_0 induced between A and B by this external current is given by

$$P_0 = i_0/(k + \tilde{k}), \tag{2}$$

where $\tilde{k}$ is the equivalent permeability of the rest of the lattice in parallel with k. The permeability $\tilde{k}$ is calculated by noting that in the uniform network, the total permeability between the two edges A and B, k_{AB}, is given by

$$k_{AB} = k_e + \tilde{k}. \tag{3}$$

Thus, we have to find k_{AB}. In order to do that we consider the situation shown in Fig. 2 again, but now with $k = k_e$. The flow field can be thought of as a superposition of two contributions: (i) the field that results when a current i_0 is injected at A and extracted uniformly at infinity; (ii) the field that results when a current i_0 is injected uniformly at infinity and extracted at B.

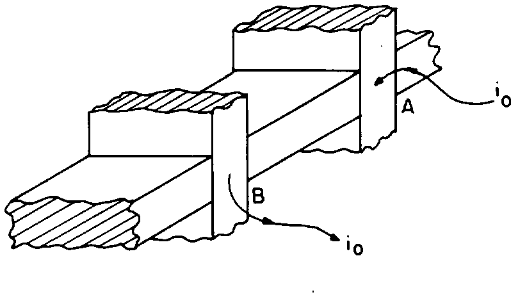

FIG. 2. The current i_0 is injected in A and B in order to compensate for the change in the local flow pattern.

If the current i_0 is injected into the volume $A(t \times t \times w)$ in such a way that the normal component of the current density is uniform over the entire surface of A, then the current that ends up flowing through the plaquette AB is simply proportional to $t \times w$. The current i_0 which is extracted from the volume $B(t \times t \times w)$ contributes an identical amount to the current through AB. Thus, we find for the total current through AB

$$i_{AB} = 2i_0[tw/(4tw + 2t^2)] = i_0/(2 + t/w) \equiv P_{AB} k_e, \tag{4}$$

where P_{AB} is the pressure drop along AB. Hence,

$$k_{AB} = k_e (2 + t/w) \tag{5}$$

and

$$\tilde{k} = k_e (1 + t/w) \equiv A k_e, \tag{6}$$

where $A \simeq 1$ when $t \ll w$. Substituting (1) into (2) and using (6) yields

$$P_0 = P_e [(k_e - k)/(A k_e + k)]. \tag{7}$$

In general, k may be chosen from a PDF, $f(k)$. In order to determine k_e, we require that the average of P_0 over all the values of k vanish, i.e.,

$$\int_0^\infty P_0(k_e, k) f(k) dk = 0. \tag{8}$$

This is an implicit equation for k_e. It can be solved exactly for some PDF's. In general, however, numerical methods must be used to obtain k_e. Note that we implicitly assumed that the resistance to flow due to edge effects is negligible. This assumption was recently justified for flow in a percolating network of pipes.[11] We further note that there is a significant difference between Eq. (8) and its analogue in Kirkpatrick's treatment.[10] This difference stems from the fact that the basic path between the neighboring sites A and B is a 1D bond, whereas we have a 2D microcrack as the basic element. Moreover, generalizing our treatment to an arbitrary dimensionality d, we will always have a d-1 dimensional crack as the basic element, while in Ref. (10) this basic element remains a 1D bond. Therefore, only in 2D do the two models coincide. The PDF of microscopic permeability always has the form

$$f(k) = (1 - p)\left[\lim_{k_1 \to 0^+} \delta(k_1 - k)\right] + p h(k), \tag{9}$$

where $\delta(k)$ is the Dirac δ function. The term that contains the δ function represents the finite probability for the absence of a crack, while $h(k)$ is the normalized PDF for the permeability of present cracks. Before we proceed to solve Eqs. (7) and (8), by considering special cases of (9), we observe that the trivial result $k_e \equiv 0$ is a solution at any concentration p (see Appendix A). Substituting (9) for $f(k)$ in (8) and assuming $k_e \neq 0$ yields an equation for the nontrivial solution for k_e,

$$\int_0^\infty \frac{k_e - k}{Ak_e + k} h(k) dk = 1 - 1/p. \tag{10}$$

The PDF of the nonzero permeability $h(k)$ can be obtained from the thickness PDF $H(t)$ by

$$h(k) = \frac{dt}{dk} H[t(k)] \tag{11}$$

whenever $t(k)$ is known. For the functional dependence of t on k we use a classical result, namely,

$$k(t) = wt^3/12\eta, \tag{12}$$

where η is the fluid viscosity coefficient. A brief derivation of this result is given in Appendix B. Thus, we find

$$h(k) = (4\eta/9w)^{1/3} k^{-2/3} H[t(k)]. \tag{13}$$

III. SOLUTIONS FOR SEVERAL PDF's

In the following we consider three different forms for the PDF $h(k_0)$, which yield exact results for k_e. The first is $h(k) = \delta(k_0 - k)$, which represents a diluted lattice in which each existing plaquette has the same fixed permeability k_0 (thickness t_0). The detailed discussion of this case serves to clarify the method for determining the percolation threshold p_c and the critical exponent, defined through $k_e \sim (p - p_c)^\zeta$ for $p \to p_c^+$.

Next we treat the case $h(k) =$ const, which corresponds to a parabolic PDF of t, $H(t) \sim t^2$. Whether this PDF is physically realizable remains to be seen. However, it readily yields an exact result. We then turn to singular power law (SPL) distributions, i.e., $h(k) \sim h^{-y}, 1 > y > 0$ for small k. Assuming $h(k)$ is such a distribution does not necessarily make $H(t)$ an SPL distribution, i.e., one having a singularity at the origin. For $0 < y \leqslant 2/3$, $h(k)$ is an SPL while $H(t)$ is regular at the origin $H(t) \sim t^x$, with $2 > x \geqslant 0$. Only for $2/3 < y < 1$ both $h(k)$ and $H(t)$ are SPL distributions.

Such distributions were shown to describe faithfully the distribution of neck widths in several models of continuum percolation.[9] Therefore, *a priori*, it seems reasonable that the distribution of crack widths in rocks may exhibit such a form. Moreover, in the context of bond conductance percolation, the critical exponent was shown to depend on the power y for such cases.[7–9] It is interesting, then, to study these PDF's also in the context of fluid flow.

A. The δ-function PDF

Assume

$$H(t) = \delta(t_0 - t),$$

which leads to

$$h(k) = \delta(k_0 - k), \tag{14}$$

where

$$k_0 = (w/12\eta) t_0^3$$

and all existing plaquettes are of thickness t_0. Equation (10) then becomes

$$(k_e - k_0)/(Ak_e + k_0) = 1 - 1/p$$

or

$$x = (1 + 1/A)\{p - [1/(A+1)]\}, \tag{15}$$

where $x = k_e/k_0$. Thus, k_e has a nontrivial solution which is linear in p and becomes negative below $p_c = 1/(A+1)$. Since k_e may not assume negative values, the correct physical solution below p_c is $k_e \equiv 0$. As usual,[10] the trivial solution also exists above p_c, but the correct solution is then the nontrivial one that tends to $k_e = k_0$ at $p = 1$. In Fig. 3 we plot $x(p)$ for $t/w = 0.01$. The exponent ζ is one as is always the case for EMT calculations with binary PDF's.[10]

B. Constant PDF

We now assume

$$h(k) = 1/k_0, \quad 0 \leqslant k \leqslant k_0.$$

Relation (10) now reads

$$1/p - 1 + \frac{1}{k_0} \int_0^{k_0} \frac{k_e - k}{Ak_0 + k} dk = 0.$$

Performing the integration we obtain

$$x \ln[1 + (1/Ax)] = S, \tag{16}$$

where

$$S = [p - 1/(A+1)]/Ap$$

and $x = k_e/k_0$. Figure 4 displays $x(p)$ for $t/w = 0.01$. Analyzing relation (16) carefully we find

$$x(p) \sim -[(p - p_c)/\ln(p - p_c)]$$

in the limit $p \to p_c^+$. Therefore, $\zeta = 1$, but there are logarithmic corrections.

C. SPL distributions

In this case we assume

$$h(k) = [(1-y)/k_0^{1-y}] k^{-y}, \tag{17}$$

$$0 \leqslant k \leqslant k_0, \quad 0 < y < 1.$$

Inserting (17) into (10) and changing variables in the integral yields

$$x^{1-y} \int_0^{x^{-(1-y)}} \frac{du}{A + u^\mu} = S, \tag{18}$$

with $\mu = 1/(1-y)$ and $x = k_e/k_0$.

We proceed to find solutions for the special cases $y = \frac{1}{2}$ and $\frac{2}{3}$.

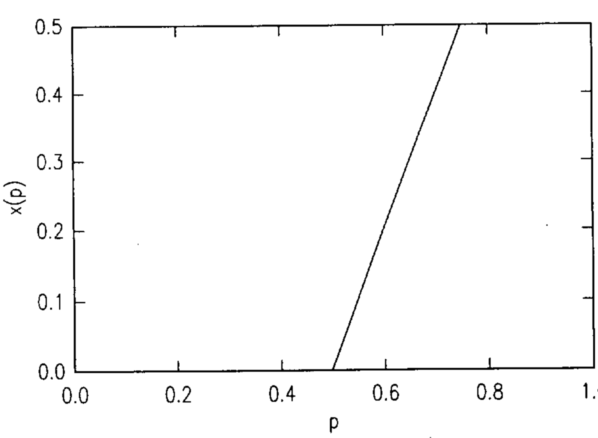

FIG. 3. $x(p)$ for the binary PDF for $t/w = 0.01$.

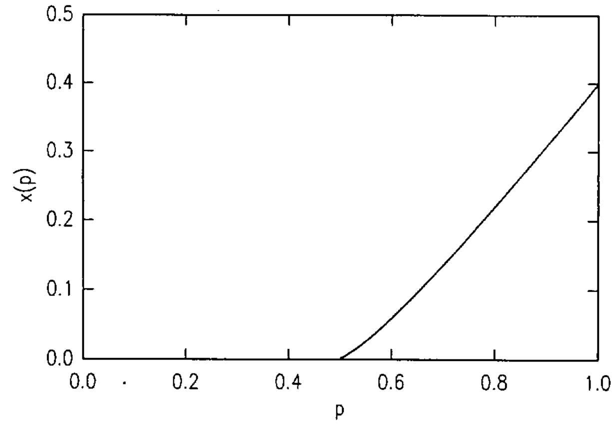

FIG. 4. $x(p)$ for the continuous PDF for $t/w = 0.01$.

1. $y = 1/2$

This value corresponds to $H(t) \sim t^{1/6}$ which is not singular at $t \to 0$. The integral in Eq. (18) can be performed yielding

$$\sqrt{Ax}\tan^{-1}(Ax)^{-1/2} = S. \quad (19)$$

The solution $x(p)$ is shown in Fig. 5 for $t/w = 0.01$. Analyzing the asymptotic behavior of x near p_c we obtain $x \sim (p - p_c)^2$, which yields $\zeta = 2$. The departure of the critical exponent from one was shown to be a general property of SPL distributions[7–9] and will be further discussed.

2. $y = 2/3$

In this case $H(t) = \text{const}$. As before, the integral in Eq. (18) can be evaluated, leading to

$$\frac{1}{3}\left(\frac{x}{A}\right)^{1/3}\left[\frac{1}{2}\ln\left(\frac{1-s+s^2}{(1+s)^2}\right) + \frac{\pi}{2\sqrt{3}} + \sqrt{3}\tan^{-1}\left(\frac{2/s-1}{\sqrt{3}}\right)\right] = \left(p - \frac{1}{A+1}\right)/Ap, \quad (20)$$

where $s^3 = Ax$. A plot of $x(p)$ for this case when $t/w = 0.01$

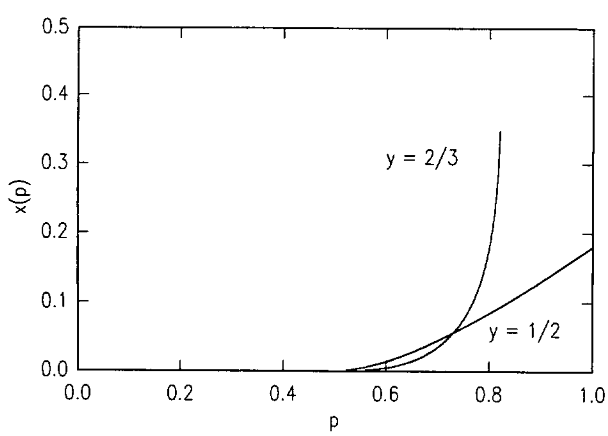

FIG. 5. $x(p)$ for two SPL distributions with $y = 1/2$ and $y = 2/3$; $t/w = 0.01$ in both cases.

is also shown in Fig. 5 for comparison with the former case. Again the critical exponent differs from one

$$\lim_{p \to p_c^+} x(p) \sim (p - p_c)^3.$$

In general, the asymptotic form for $x(p)$ when $h(k)$ is given by Eq. (17) is

$$x(p) \sim (p - p_c)^{1/(1-y)}, \quad p \to p_c^+. \quad (21)$$

Heuristic, as well as semirigorous, arguments have been advanced to explain this behavior in the cases of electrical conductivity.[7–9] In order to obtain the general result (21) we observe that in Eq. (18) when $p \to p_c^+ = 1/(A+1)$, $x \to 0^+$, and therefore the upper limit of integration tends to $+\infty$. Since $\mu > 1$, the integral converges to a finite number, yielding (21).

IV. CONCLUSIONS AND DISCUSSION

We treated the problem of plaquette percolation as a model for fluid flow in a randomly porous medium, and solved it using the EMT approximation. Our main results can be summarized as follows: (1) We found the threshold concentration p_c, which turns out to be considerably higher for this problem than for the usual bond percolation; (2) we solved explicitly for the bulk effective permeability as a function of the concentration of existing cracks. This was done assuming a number of different probability density functions for the thickness of the cracks; (3) The critical exponent ζ was found to be one for the binary and the constant distributions. In the latter case, however, there is a logarithmic correction. For singular power law distributions, $h(k) \sim k^{-y}, 1 > y > 0$, we found $\zeta = 1/(1-y)$, in agreement with previous results.[7–9]

Result (1) stems mainly from the restriction that flow between neighboring plaquettes occurs only if they have a common edge. In previous (numerical) studies of the flow problem,[3] flow (or conduction) between plaquettes was assumed to occur even if they had only one common vertex, which results in a lowered p_c. The treatment we presented can be generalized to higher dimensions d. We note that the parameter A in Eq. (6) depends on the dimensionality of the small volume between connected hypercracks. Thus, for general d we would have $A = d - 2 + t/w$, and the threshold becomes

$$p_c = 1/(d - 1 + t/w). \quad (22)$$

The dependence of the permeability k on crack thickness t will also change when $d \neq 3$, so that corresponding changes can be expected in some of the values found for the critical exponent. The actual values found for p_c and for ζ are not expected to be exact since the EMT yields incorrect values for $d > 1$. However, the effect of various PDF's on the values of ζ is expected to be at least qualitatively reliable.

Let us consider the quasi-invariant B_c discussed in the literature[12–14]

$$B_c = Zp_c, \quad (23)$$

where Z is the number of nearest neighbors. Substituting (22) into (23), and recalling that in our hypercubic lattice $Z = 2d$, yields

$$B_c \simeq 2d/(d-1)\{1-[t/w(d-1)]\}. \quad (24)$$

To zero order in t/w this relation curiously coincides with a long known conjecture[12] advanced for percolation of thin disks in 2D and for spheres in 3D. In our case, however, B_c is not a real invariant but depends weakly on the properties of the system (t and w).

We found the explicit functional form of the effective permeability for several model PDF's. It would be interesting to find this form using a realistic or measured distribution of the crack thickness in Eq. (10). The result of such a calculation can then be compared with experimental data, thus testing the applicability of the EMT to this problem. We were unable to find such a measured distribution in the literature and therefore could not examine this interesting question.

Turning to the critical exponent ζ, we first note that a general analysis shows that ζ is always one if $h(k)$ vanishes sufficiently rapidly as $k \to 0$, so that the integral

$$I = \int_0^\infty \frac{h(k)dk}{Ak_e + k}$$

is finite even for $k_e = 0$. This assertion follows immediately if we rewrite (10) as

$$k_e = (p - p_c)/ApI, \quad p_c = 1/(A+1).$$

The departure of ζ from unity for SPL distributions is in accordance with previous results for the linear[7–9] conductivity.

As mentioned in the text, an SPL distribution $h(k)$ corresponds, for $0 < y \leqslant 2/3$, to a regular power law PDF of t, $H(t) \sim t^x$ with $2 > x \geqslant 0$. Thus, in this range of values of y, we demonstrated that even regular power law PDF's of crack thickness may result in a nonuniversal behavior (i.e., ζ that is dependent on the distribution), leading to an anomalous critical exponent.

The plaquette percolation model which we have treated is also valid for the effective conductivity of metal plates embedded in an insulating host. Hence, all our results concerning different PDF's hold for that problem as well, including the values of ζ and p_c in the various cases. Therefore, if such a system could be realized, it would provide another test for the applicability of this approach.

ACKNOWLEDGMENTS

This research was supported in part by grants from the Israel Academy of Sciences and Humanities, from the Soreq Nuclear Research Center of the Israel Atomic Energy Commission, and from the U.S.-Israel Binational Science Foundation.

APPENDIX A

In order to demonstrate that Eq. (9) in the text always has the trivial solution $k_e = 0$, let us consider for simplicity the binary distribution,

$$f(k) = (1-p)\delta(k-k_1) + p\delta(k-k_2). \quad (A1)$$

Substituting it into Eq. (8) in the text yields

$$(1-p)[(k_e - k_1)/(Ak_e + k_1)] + p[(k_e - k_2)/(Ak_e + k_2)] = 0. \quad (A2)$$

This is a quadratic equation and has the two solutions

$$k_e = (\Phi \pm \sqrt{\Phi^2 + 4Ak_1k_2})/2A, \quad (A3)$$

where

$$\Phi = p(A+1)(k_2 - k_1) - (k_2 - Ak_1).$$

Inserting $k_1 = 0$ in (A3) yields a trivial solution $k_e = 0$ and a nontrivial one $k_e = \Phi/A$. However, if one tries to approach the limit $k_1 \to 0^+$ directly in (A2), one has to be careful in order not to lose the trivial solution. Keeping this in mind, let us consider Eq. (8) with a PDF $h(k)$ that vanishes for $k < k_2$, where $k_2 > k_1 > 0$:

$$(1-p)\frac{k_e - k_1}{Ak_e + k_1} + p\int_{k_2}^\infty \frac{k_e - k}{Ak_e + k} h(k)dk = 0. \quad (A4)$$

When both k_1 and k_2 are small, (A4) always has a solution for k_e that is also small. For example, $k_1 < k_e < k_2$ when p is small and $-k_2/A < k_e < -k_1/A$ when p is large (close to 1). We can now let both k_1 and k_2 approach zero and let $h(k)$ tend to the actual PDF. Obviously, the solution for k_e also approaches zero.

APPENDIX B

Each plaquette represents a crack between two parallel square plates (Fig. 6) through which the fluid flows in the y direction. We assume that the edges do not affect the permeability. We also assume that no current flows in the x and z directions. A force $F(z)$ is exerted on the fluid only due to its viscosity η. We can thus write the following differential equation for the y component of the fluid velocity in the crack v

$$F(z) = -\frac{dv}{dz} wL\eta = wz\,\Delta P, \quad (B1)$$

where ΔP is the pressure drop along the crack. Integrating (B1), we obtain

$$v = -(\Delta P/2L\eta)[z^2 - (t/2)^2], \quad (B2)$$

where the vanishing of v at $z = \pm t/2$ was assumed as a boundary condition. In order to find the total flow Q through the plaquette we integrate this velocity field

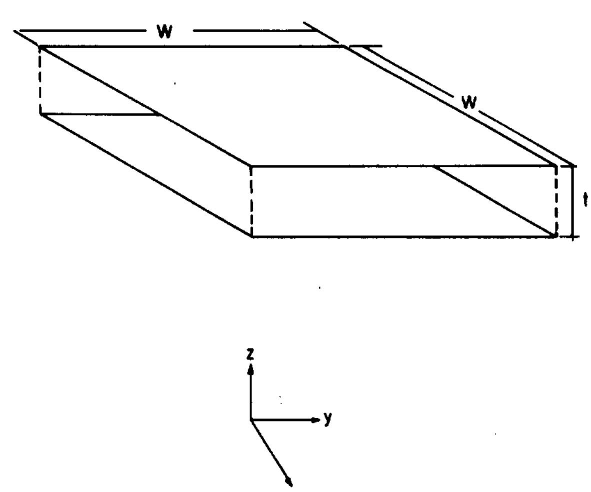

FIG. 6. Flow between two parallel infinite plates.

$$Q = w \int_{-t/2}^{t/2} v\, dz = \frac{wt^3}{12\eta} \frac{\Delta P}{L}. \tag{B3}$$

Relation (12) in the text then follows.

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