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Granular matter and the marginal rigidity state

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Abstract

Model experiments are reported on the build-up of granular piles in two dimensions. These show that, as the initial density of falling grains is increased, the resulting pile has *decreasing* final density and its coordination number approaches the low value predicted for the theoretical marginal rigidity state. This provides the first direct experimental evidence for this state of granular matter. We trace the decrease in the coordination number to the dynamics within an advancing yield front between the consolidated pile and the falling grains. We show that the front's size diverges as the marginal rigidity state is approached, suggesting a critical phenomenon.

(Some figures in this article are in colour only in the electronic version)

Recent theoretical works [1–5] have suggested that a distinctive, marginal rigidity state exists for rigid cohesionless grains. This state is characterized by low connectivity and exhibits stress transmission governed by its geometry alone. It has yet to be established whether this applies to real granular matter, such as sand or grain, but it is a candidate to underpin recent success in modelling macroscopic stress transmission [6, 7]. Here we present the first direct experimental evidence, from idealized granular systems in two dimensions as shown in figure 1, that the marginal solid state of matter does indeed exist. Our experiments also reveal a diverging lengthscale, suggesting that the state is a true critical phenomenon.

The key idea behind this state is that rigid grains falling to form a pile will rearrange and consolidate, until their mean coordination number z reaches a critical value z_c at which the net force and torque on each grain can first be balanced by the intergranular forces. At this marginal rigidity state the forces, and hence the transmission of stress, are exactly determined by balance alone, without reference to any internal constitutive behaviour of the individual grains. This state separates fluid from conventional solid: so long as $z < z_c$, mechanical balance cannot in general be obtained and the system is fluid. By contrast, when $z > z_c$ the system can be mechanically balanced, but additional information is needed to determine the intergranular forces. The extra information usually comes from constitutive stress–strain relations and consistency of the strain field, leading to stress transmission typical of traditional solids.

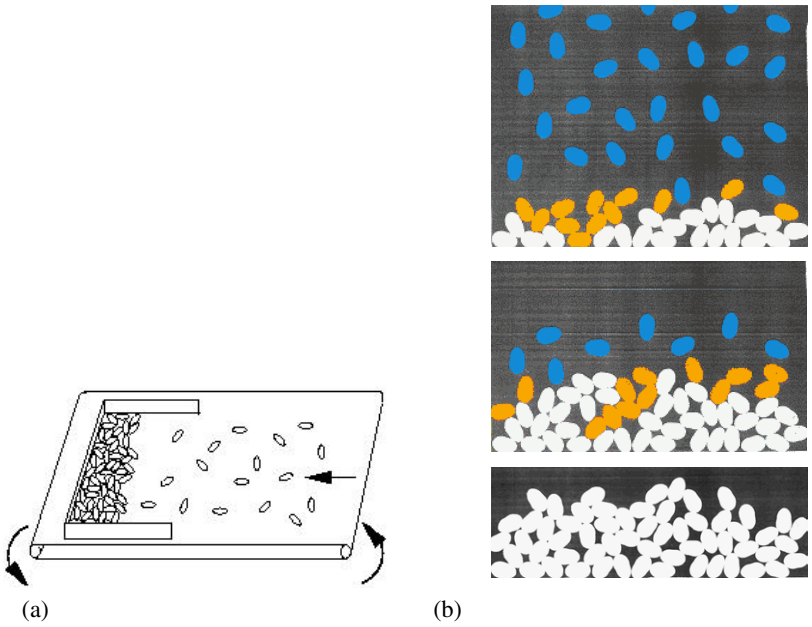

Figure 1. The experiment. (a) Sketch of our experimental setup: cardboard grains are conveyed by a moving surface towards a stationary ⊔-shaped collector of similar material. The grains are effectively rigid, with high mutual friction, and the slow advance rate minimizes inertial effects. (b) The growth of a pile, with time running from top image to bottom. The grains coloured darker grey (blue) are in an analogue of free fall, comoving with the base towards the growing pile. The white grains have come permanently to rest, and friction relative to the base supplies an analogue of gravitational force. The grains coloured light grey (orange) have not fully consolidated (see also figure 3).

For convex grains with friction the critical coordination number predicted is notably low: $z_c = 3$ and 4 in two and three dimensions, respectively [4]. The corresponding values for sequential packing are 4 and 6, and regular sphere close packing gives even higher values of 6 and 12, respectively. Thus coordination number is a key indicator of the marginal rigidity state. Traditional studies where particles fall one at a time [8] cannot address this issue, which originates in the cooperative dynamics of the particles, as we show here.

Our experiments are illustrated in figure 1. In an analogue of free fall, grains were conveyed by the moving base until they encountered either the collecting boundary or its accumulating pile. We have used model noncircular grains of approximate area $1.5 \pm 0.2\,\mathrm{cm}^2$. The particles were cut from 1.6 ± 0.2 mm cardboard using a stainless steel punch. The fibrous nature of the cardboard ensured that the cut edges gave a high coefficient of intergranular friction, which we estimated from slipping tests to be at least 50. The particles were not entirely convex, and the implications of this will be discussed below. The experiment was carried out on a horizontal glass plate, with the pile building up within a ⊔-shaped collector surface of similar material (see figure 1) and internal dimensions approximately $18 \times 24\,\mathrm{cm}^2$. The grains were initially placed on a thin transparent film lying on top of the plate at a notionally random distribution subject to the requirement of no contacts and reasonably uniform density.

The 'free falling' grains were conveyed towards the collector at an approximate speed of 0.5 m min^{-1} (maximum): at high initial densities this was done by moving the collector towards the grains. At low initial densities the film was moved towards the stationary collector, and the 'initial' distribution of grains was created only as it approached the pile. Winding

mechanisms proved vulnerable to stick–slip motion of the film and were rejected in favour of driving the experiment directly by hand: care was taken to maintain a constant advance rate and particularly to avoid relative transverse motion between the collector surface and the free falling grains. The slow fall rate ensured both negligible grain deformation and minimization of inertial effects. The force of sliding friction relative to the moving base played a role analogous to gravitational force on a stationary mass.

The data were accumulated in the form of images of the growing pile, taken at regular intervals during the process and when all consolidation had ceased. For each final pile we have counted its grains, the boundary grains, and the total contacts (including double contacts counted twice). The quantitative determination of the number of contacts was done as follows: for every near (or apparently actual) contact (including to the boundary) we measured the apparent spacing to an accuracy of $1/7$ mm. Graphs of the number of contacts closer than x exhibited linear behaviour for distances between our resolution and the spacing of typical non-contacting neighbours, and extrapolating this back to zero separation gave us the number of true contacts. The result of this systematic calculation agreed very well with an initial straightforward, but subjective, counting of contacts by eye (as reported in [11]). The separate count of just double contacts was made by eye.

Because the theoretical predictions of coordination number are based fundamentally on the expectation of three constraints per grain at the marginal state, we measured the coordination number as $2\times$ (total number of contacts)/(total number of grains); if calculating coordination from individual grains, this means double counting coordinations from a grain to the boundary. The (relatively few) double contacts were counted as two separate contacts.

Figure 2 (top) shows how the final density ρ_{p} of the resulting pile *decreases* with increasing the initial density ρ_{i} at which grains were fed in. We believe this decrease may be characteristic of a jamming system, and our discussion of the front below offers some insight as to how this comes about. By extrapolating a fit of the pile density, we find that it would match the density of the feed at $\rho_{\mathrm{c}} = 0.465(5)$. This is the natural limiting density of our experimental series. The value of ρ_{c} depends on the particular fit. However as we discuss below, what is crucial is not the exact value, but rather the interpretation of this density and the behaviour of the system in its neighbourhood.

Figure 2 (bottom) shows how the coordination number z varies with initial density ρ_{i} in our experiments. The striking feature is that as ρ_{i} increases, z decreases towards an extrapolated value $z(\rho_{\mathrm{c}}) = 3.1$, in qualitative agreement with the simple prediction $z_{\mathrm{c}} = 3$ for marginal rigidity. We know of no other interpretation of how starting at higher density leads to lower coordination number of this value.

Due to their shape, neighbouring grains are *just* capable of making contact at two points, which we term double contacts. This complication is not addressed by the theory for convex grains. We have counted these as two contacts each, but we note that a double contact imposes only *three* constraints in total, not four as two separate contacts would (see discussion below): one constraint is thus 'wasted' per double contact. Counting the coordination number as we have done, this means that the theoretically predicted critical coordination number z_{c} should be shifted (in two dimensions) to

$$z_{\mathrm{c}} = 3 + w_2, \tag{1}$$

where w_2 is (without any double counting) the number of double contacts per grain. From our experiments we measured $w_2 \approx 0.1$ with no significant dependence on ρ_{i}, leading to an excellent quantitative agreement of z_{c} with the extrapolated value $z(\rho_{\mathrm{c}})$.

At lower initial densities the measured coordination number rises, approaching $z_0 = 3.7$ as ρ_{i} tends to zero. This is in fair agreement with the value $z = 4$ expected for sequential

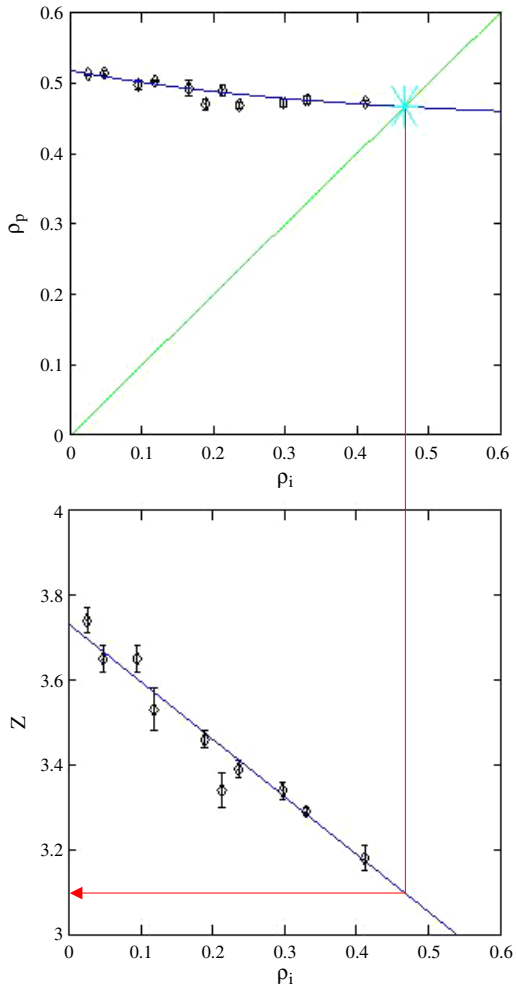

Figure 2. The behaviour of the consolidated pile. Top: the final density of the consolidated pile, ρ_{p}, as a function of the initial density of the free falling grains, ρ_{i}. We suggest the decreasing behaviour is characteristic of the jamming nature of our system. The line interpolating the experimental points is a numerical fit $\rho_{\mathrm{p}} = 0.446 + 0.072\ \mathrm{e}^{-2.7\rho_{\mathrm{i}}}$, to guide the eye and to enable identify the point circled where the initial and final densities extrapolate to be equal, $\rho_{\mathrm{c}} = 0.465(5)$. This is the natural limiting density of our experiments. Bottom: the mean coordination number z of each consolidated pile as a function of the initial density, ρ_{i}. The data extrapolate to $z = 3.1(1)$ at the limiting density ρ_{c} (vertical line), in good agreement with prediction for the marginal rigidity state. The coordination number approaches $z = 3.7$ in the zero density limit, which approximates to sequential packing. In both figures 2 and 3 the error bars reflect one standard error in the mean from several repetitions of the experiment.

packing, where each arriving grain would come fully to rest with two contacts before another encounters the pile. This can be viewed in terms of wasted constraints as follows: when each grain joins an already rigid pile just three degrees of freedom (two translations plus one rotation) are lost, whereas two contacts generally imply four constraints. The result is that one constraint is 'wasted' and correspondingly there is one degree of freedom in the intergranular forces which cannot be determined from the geometry alone. This is in precise analogy to

our discussion of double contacts above. In the experiments, we observe that occasionally an incoming grain will dislodge other previously stationary grains before settling down. Thus the positioning is not strictly sequential, and this explains why the coordination number does not reach as high as 4.

At intermediate densities, the coordination numbers can be interpreted in terms of intermediate levels of constraint wastage. A constraint is wasted every time a loop with only one hinge is closed, and every wasted constraint introduces an ambiguity regarding the transmission of forces in the pile. A double contact is special in this respect, because its force ambiguity is localized within the closed doublet of particles and therefore does not affect macroscopic stress transmission. Precisely which wasted constraints preserve marginal rigidity in this way is an open question.

While the above results verify the existence of the marginal rigidity state, our experiments can also be interpreted on a macroscopic level in terms of a yield front that propagates ahead of the consolidated pile. We define the front, at any given moment, as all the grains that have already collided with the pile (and its connected front) but have not yet reached their final position in the consolidated configuration. The photographs taken during each piling process were used to identify the yield front by comparing the intermediate positions of the grains to their position in the final pile. The comparison was made by superposing the two photographs on top of a lightbox and determining overlaps by eye. For front measurement only, we conducted a second series of experiments on elliptic grains of half the original area and a similar aspect ratio. These piles, which were significantly deeper relative to their grain size, were prone to buckling under load, which we suppressed by adding a roof to the collector with 'ground' clearance twice the grain thickness.

Figure 1(b) shows three stages in the time evolution of a pile with its front coloured light grey (orange). Figure 3(a) shows the front for three different values of ρ_{i}, and it can be clearly seen that the higher the initial density the deeper the front. In the sequential packing regime, $\rho_{\mathrm{i}} \to 0$, the front is less than a monolayer, and each falling grain collides with an almost completely rigid pile; hence the significant constraint wastage and high coordination number. At higher ρ_{i}, falling grains encounter a layer that is still rearranging; then the constraints from their contacts are less likely to be wasted, resulting in a lower final coordination number.

We characterized the front by its number of particles per unit width of the base, M, and we report this in terms of the dimensionless combination $m = M/\rho_{\mathrm{p}1/2}$. The increase of m with $\rho_{\mathrm{i}}/\rho_{\mathrm{p}}$ is plotted in figure 3(b). On the basis of these data and a simple theory presented below, we conjecture that the front size in an infinite system *diverges* as $\Delta\rho = (\rho_{\mathrm{p}} - \rho_{\mathrm{i}})/(\rho_{\mathrm{p}} + \rho_{\mathrm{i}}) \to 0$. In our main series of experiments the front size approached our (limited) system size by $\rho_{\mathrm{i}}/\rho_{\mathrm{p}} = 0.6$, so the data shown have been supplemented by further measurements with a larger sample of smaller grains.

We can estimate theoretically the divergence of the front depth ξ as $\Delta\rho \to 0$ by considering the time available for grains to rearrange whilst in the front. From conservation of grains it follows that the mean velocity of grains relative to a frame of reference moving with the centre of mass of the front is approximately $v_{\mathrm{r}} = v_0/(2\Delta\rho)$, where v_0 is the speed of the falling grains relative to the pile. The *time* a grain resides in the front is then estimated by $\tau \approx \xi/v_{\mathrm{r}}$. During this time the maximum sideways displacement that a grain can achieve is of order $\Delta r \approx \tau v_0$, in terms of which we can now express the front width as

$$\xi \approx \Delta r/(2\Delta\rho). \tag{2}$$

The piles (before and after consolidation) are far from close-packed; therefore it takes relative motion of order of a particle size to achieve new contacts. Thus, we expect Δr to be of order of a particle size for reorganization of the grain distribution to be achieved from the feed into

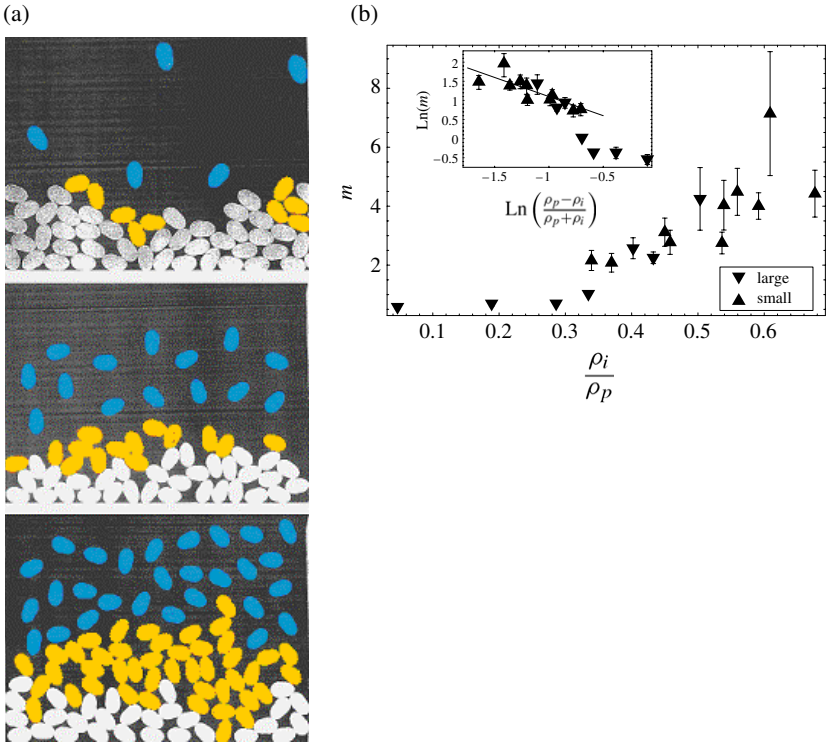

Figure 3. The behaviour of the consolidation front. (a) The front (coloured red) is defined as those grains which have encountered the pile but not yet reached their final consolidated positions, shown here from three different experiments at initial densities 0.024 (top), 0.142 (middle), and 0.236 (bottom). At the low density the front is less than one grain deep, while at the high density it is more than half the pile and the consolidation is highly cooperative. (b) The measured dimensionless front mass m, where $m\rho_{\mathrm{p}1/2}$ is the number of grains per unit of base, is shown as a function of the relative initial density $\rho_{\mathrm{i}}/\rho_{\mathrm{p}}$. This shows a diverging trend, with reasonable overlap between data as illustrated left and a second series of front measurements taken with more and smaller grains. The inset shows m plotted against $\Delta\rho = (\rho_{\mathrm{p}} - \rho_{\mathrm{i}})/(\rho_{\mathrm{p}} + \rho_{\mathrm{i}})$ on logarithmic scales, which a simple analysis (see the text) would predict to exhibit a critical divergence $m \propto \Delta\rho^{-1}$, as shown by the guideline for comparison.

the pile. Certainly by experimental observation we are confident that if $\Delta r \to 0$ as $\Delta\rho \to 0$, then it does so only very slowly. Hence we are led to the front width diverging as

$$\xi \propto \Delta\rho^{-\nu} \tag{3}$$

with $\nu = 1$ if Δr is set by the grain size. Due to the limited size of systems available to us it is difficult to judge how well this value describes our data, and additional experiments on larger systems are needed for this.

The very existence of a diverging front size as the marginally rigid state is approached strongly suggests that the latter is a continuous critical point, analogous to various self-organizing critical systems. If so the exponent ν should be insensitive to details of the grains, corresponding to a new universality class, but it should be sensitive to the two-dimensional nature of our experiments and to the dominance of friction over inertia and hydrodynamics. At the critical point the entire system is marginally rigid, transmitting stress as an isostatic system. As the number of contacts increases, small 'normal' regions form that are no longer isostatic, and the system moves away from the critical point. This critical point is not related

to the roughness exponent observed in numerical deposition processes, which are generically in the limit $\rho_i \to 0$.

The transition from a yielding layer to a rigid pile bears a resemblance to the rigidity percolation problem. However, there are several fundamental differences: (1) in the percolation models, bonds are created one at a time and randomly, irrespective of the structure; (2) percolation bonds can support tensile, as well as compressive, forces; (3) once formed, percolation bonds cannot be broken. By contrast, in our experiment, contacts are constantly made and broken, they can support only compressive forces, and, very significantly, the evolution of the contact network is cooperative in response to local forces and the dynamics of the consolidation process.

A possibly related transition has been observed experimentally when a granular system under shear jams as the density reaches a critical value [9]. Unfortunately, the coordination number was not monitored in that experiment, and it is difficult to judge whether in that experiment the system was marginally rigid or not.

We believe that three-dimensional experiments would give similar results, and the classic experiment of Bernal and Mason [10] should be noted. They have found that loose packing of ball bearings within a rolled cylinder with dimpled sides gave $z = 5.5$, a value which is intermediate between the marginal rigidity values $z_c = 4$ with friction and $z_c = 6$ for frictionless spheres, and below that of sequential packing, $z = 6$. While to measure the coordination number *in situ* in three dimensions is harder, the front depth may be easier to measure via the density profile.

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