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Dynamics of twists on antiferromagnetic spin chains: Theory

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Abstract. The equation of motion of twists on classical antiferromagnetic Heisenberg spin chains are derived. It is shown that twists interact *via* position- and velocity-dependent long-range two-body forces. A quiescent regime is identified wherein the system conserves momentum. With increasing kinetic energy the system exits this regime and momentum conservation is violated due to walls annihilation. A bitwist system is shown to be integrable and its exact solution is analysed. Many-twist systems are discussed and novel periodic twist lattice solutions are found on closed chains. The stability of these solutions is discussed.

PACS. 75.60.Ch Domain walls and domain structure – 05.45.-a Nonlinear dynamics and nonlinear dynamical systems

1 Introduction

Classical antiferromagnetic spin chains are far from being fully understood [1]. At low energies, it has been known for quite some time that they support spin wave and instanton solutions [2,3], but recently it was found that they also admit multitwist solutions [4,5]. The aim of this paper is to study the kinetics and dynamics of these multitwists. In the following I introduce these solutions, derive their equations of motion, and discuss some of their intriguing properties. In particular, I identify a 'quiescent' regime where the system conserves momentum. Outside this regime twists collide and annihilate, whereupon the momentum of the system changes discontinuously. I show that a system of two twists on an open-ended chain is integrable and its exact solution is found and analysed. Finally, I discuss multitwists on closed chains and argue that the twists form a periodic lattice. These lattices are found to be stable against perturbations of wavelength longer than the lattice periodicity and unstable otherwise.

2 Multitwist solutions: Brief review

Assume a chain of antiferromagnetically coupled spins with only nearest neighbour interactions described by the spin exchange Hamiltonian

$$H = \sum_i J\mathbf{S}_i \cdot \mathbf{S}_{i+1} \qquad J > 0. \tag{1}$$

Distinguishing between the odd ($\mathbf{S}_o$) and even ($\mathbf{S}_e$) sublattices, one defines the total magnetisation, $\boldsymbol{\zeta} = A(\mathbf{S}_e + \mathbf{S}_o)$, and the staggered magnetisation, $\boldsymbol{\eta} = B(\mathbf{S}_e - \mathbf{S}_o)$. A and B are normalisation constants. The equation of motion of the spins can be converted into equations for these fields. In the low energy limit, and after rescaling time and position appropriately, one finds that the lowest order equation with a nontrivial solution is the one for the staggered magnetisation [6,7],

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \boldsymbol{\eta} \times \frac{\partial \boldsymbol{\eta}}{\partial x} \cdot \tag{2}$$

The unit vector $\boldsymbol{\eta}$ is characterised by its angles, θ and ϕ, whose equations of motion can be obtained from (2). Defining $\cos\theta = \tanh\psi$ [4,5] simplifies those equations to

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial t}; \quad \frac{\partial \psi}{\partial t} = -\frac{\partial \phi}{\partial x} \tag{3}$$

which are the familiar Cauchy-Riemmann equations. Charge-like sources correspond in this context to spins fixed at a given orientation and lead to logarithmic singularities. Beyond sources and sinks all the analytic functions solve (3). Once these solutions are translated back into the $\theta - \phi$ language, the sources correspond to the well known n-instanton solutions discussed by Belavin and Polyakov [3,8]. The analytic functions give rise to travelling multitwists whose locations along the chain correspond to the nodes of ψ. Each such twist separates between a state of $\cos\theta = 1$ and $\cos\theta = -1$. It is these solutions on which I focus here.

3 The equations of motion

Assume an open-ended chain with N twists on it. The general solution for ψ can be represented in the form

$$\psi = \sum_{n=1}^{N} P_{N-n}(t)x^n = \prod_{n=1}^{N}(x - x_n(t)) \tag{4}$$

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where P_n are polynomials of order n and a twist in the staggered magnetisation corresponds to a real value of x_n. The coupled field ϕ has a similar form, but is of little interest for the purpose of this discussion.

Assuming no sources, ψ must obey Laplace's equation

$$\nabla^2\psi = \psi\left[\sum_{n=1}^{N}\sum_{m\neq n}^{N}\frac{\dot{x}_n\dot{x}_m+1}{(x-x_n)(x-x_m)}-\sum_{n=1}^{N}\frac{\ddot{x}_n}{x-x_n}\right]=0. \tag{5}$$

Since this equation is valid for all x it must hold in the vicinity of x_n and therefore

$$\ddot{x}_n - 2\sum_{m\neq n}\frac{1+\dot{x}_n\dot{x}_m}{x_n-x_m}=0 \tag{6}$$

which is the equation of motion of the nth twist. It describes a 'force' on it due to interaction with the other twists. The following should be noted: (a) The force on the nth twist consists of a *sum of two-body forces* between it and every other twist. There are no forces involving more than two twists. (b) Newton's third law of action and reaction applies in that the force on the nth twist due to twist m is equal but opposite to the force by twist n on m. (c) Equations (6) are *invariant under time reversal*. For any forward solution $\psi_f(t)$ thre exists a backward solution, $\psi_b(t)=\psi_f(-t)$. (d) The system *conserves momentum*, as can be verified by summing over n and noting that the interaction terms cancel in pairs. It follows that

$$\ddot{\gamma}=\frac{1}{N}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_{n=1}^{N}x_n=0 \tag{7}$$

leading to $\dot{\gamma}=\gamma_0$ being a constant of the motion. Since γ is the centre of mass of the system of twists then the above amounts to conservation of the total momentum. (e) The interactions between the twists are long-ranged and consist of two contributions: a Coulomb-like repelling term and a coupling between the velocities of the twists. This interaction will be shown to give rise to rich dynamics.

4 The Bitwist: An exact integrable solution

For insight into the dynamics of twists it is best to first look into the kinematics of a small system: a chain supporting a double twist. Starting from the equations of motion of x_1 and x_2, I define two new variables: the centre of mass γ and the separation $\delta=(x_2-x_1)/2$. Using the fact that $\dot{\gamma}=p_0$ is constant, a straightforward manipulation gives

$$\delta\ddot{\delta}+\dot{\delta}^2-c^2=0 \tag{8}$$

where $c=\sqrt{(1+p_0^2)}$. This equation is invariant under $\delta\to-\delta$ as well as under time reversal. It admits a linear solution $\delta_1=\pm ct+\delta_0$, with δ_0 an initial separation and the $\pm$ corresponding to either a forward or backward solution. Equation (8) has an interesting first integral and constant of the motion,

$$\delta^2(\dot{\delta}^2-c^2)=\text{const.} \tag{9}$$

The linear solution is a member of a larger family:

$$\delta=\pm\sqrt{c^2t^2+2v_0\delta_0t+\delta_0^2}. \tag{10}$$

Here δ_0 and v_0 are the initial separation and its rate of increase, respectively. It can be observed that asymptotically the rate of increase tends to a constant, $\dot{\delta}=c$, which depends only on the initial momentum of the system, p_0. It is straightforward to interpret this solution: Starting with $v_0>0$, the separation between the two walls increases forever. When $0>v_0>-c$ the walls approach each other at an ever slower pace, until at time $t=\mid v_0\mid\delta_0/c^2$ they achieve a minimum separation of $\delta_{min}=\delta_0(1-v_0^2/c^2)$ and then they move apart again. Finally, when $v_0<-c$ the gap between the twists decreases until, at time $t_c=\delta_0\left(|v_0|-\sqrt{v_0^2-c^2}\right)/c^2$, they collide. Upon collision the state between the twists disappears altogether, which means that the twists annihilate each other.

5 Multitwist systems and stripe solutions

Turning to many-twist systems, I would like to focus attention on a 'quiescent' regime, where all nearest neighbour twists obey the condition $\dot{x}_n\dot{x}_{n+1}>-1$. In this regime all twists repel, leading to an ever expanding system. Since no collisions can occur the number of twists is unchanged and the system conserves momentum. Outside the quiescent regime there appear attractive forces between nearest neighbour and collisions can happen, whereupon twists annihilate. On annihilation the total momentum, as defined above, jumps discontinuously since the annihilated twists take their momentum out of the system. Thus, momentum is conserved only between annihilation events. Interestingly, twists may also be created in pairs [9], which would similarly change the total momentum. Numerical studies of multitwists on open-ended chains, which cannot be included for lack of room, support the analysis presented here.

Finally, a fascinating variation on the theme is the behaviour of N twists on a closed chain. Equations (6) admit a periodic lattice solution with twists positioned along a L-long chain at intervals of L/N. These can be regarded as alternating stripes of the two states with the twists separating them, and are therefore referred as stripe solutions. These solutions can be analysed by first expressing the field ψ in the form

$$\psi=\mathrm{Re}\sum_{n=0}^{N}b_n\left(\frac{t\mathrm{e}^{i\theta}}{t_0}\right)^n=\psi_0t^N\prod_{n=1}^{N}(\mu-\mu_n). \tag{11}$$

Here, N and the coefficients b_n are determined by initial data at some time t_0, $\mu=\cos\theta$, $\mu_n=\cos\theta_n$, and

ψ_0 is a constant. The equations of motion of the walls are obtained by substituting the rightmost expression into $\nabla^2\psi = 0$ in cylindrical coordinates. This gives the equation of motion of the nth domain wall at θ_n

$$\ddot{\mu}_n + 2N\dot{\mu}_n + \mu_n = 2\sum_{m\neq n}^{N} \frac{\dot{\mu}_n\dot{\mu}_m + 1 - \mu_n^2}{\mu_n - \mu_m} \tag{12}$$

where a dot stands now for a derivative with respect to $\tau = \ln(t - t_0)$. This equation describes quite intricate dynamics. Nevertheless, it can be seen from equation (12) that eventually the solution settles into the highest mode, $\psi \to b_N(t/t_0)^N \cos(N\theta)$, and therefore the dynamics that equation (12) gives rise to, while rich and interesting, must be transient.

The stability of the stripe structure depends very much on the type of perturbation to it. From equation (12) we can see that perturbations of wavelength longer than L/N grow slower than the Nth mode and therefore would be damped out. However, wavelengths shorter than that would grow *faster* than t^N and would eventually take over. For example, consider a perturbation at $t = t_1 > t_0$ by a term of the form $b_M \cos(M\theta)$, where $M > N$. Presuming a small perturbation means $b_M \ll b_N$. The time that it takes this term to destabilise the N-period solution, t_d, can be estimated from the condition

$$b_N \left(\frac{t_d}{t_0}\right)^N \approx b_M \left(\frac{t_d - t_1}{t_1}\right)^M . \tag{13}$$

From this expression it is straightforward to see that $t_d \gg 2t_1$, indicating that even though the stripe lattice is in principle unstable, it would take a long time before it actually deteriorates beyond recognition.

In conclusion, I have discussed the dynamics of twists on classical antiferromagnetic Heisenberg chains. Their explicit equation of motion has been derived and they have been shown to experience only two-body, position- and velocity-dependent, forces. The forces are long ranged and can be either repulsive, in a quiescent regime, or attractive. The dynamics is rich both due to the highly nonlinear nature of the equation of motion and because twists annihilate upon collision. An explicit solution has been presented for a bitwist system. On ring-like chains the equation of motion has been shown to give rise to periodic solutions. These solutions have been found to be stable against perturbations of long wavelength but unstable to short ones.

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