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Auxetic strains—insight from iso-auxetic materials

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Auxeticity is the result of internal structural degrees of freedom that get in the way of affine deformations. This paper proposes a new understanding of strains in disordered auxetic materials. A class of iso-auxetic structures is identified, which are auxetic structures that are also isostatic, and these are distinguished from conventional elasto-auxetic materials. It is then argued that the mechanisms that give rise to auxeticity are the same in both classes of materials and the implications of this observation on the equations that govern the strain are explored. Next, the compatibility conditions of Saint Venant are demonstrated to be irrelevant for the determination of stresses in iso-auxetic materials, which are governed by balance conditions alone. This leads to the conclusion that elasticity theory is not essential for the general description of auxetic behaviour. One consequence of this is that characterisation in terms of negative Poisson's ratio may be of limited utility.

A new equation is then proposed for the dependence of the strain on local rotational and expansive fields. Central to the characterization of the geometry of the structure, to the iso-auxetic stress field equations, and to the strain-rotation relation is a specific fabric tensor. This tensor is defined here explicitly for two-dimensional systems, however, disordered. It is argued that, while the proposed dependence of the strain on the local rotational and expansive fields is common to all auxetic materials, iso-auxetic and elasto-auxetic materials may exhibit significantly different macroscopic behaviours.

Keywords: Iso-auxetic; Elasto-auxetic; Fabric tensor; Disordered structures; Stress–strain

1. Introduction

Auxetic materials expand laterally when stretched and contract when compressed, a phenomenon that is commonly regarded as a display of a negative Poisson's ratio. This behaviour originates from particular structural characteristics on the cellular level that give rise to the unfolding of basic structural elements upon stretching and folding back in upon compression. Cellular solids that exhibit such a behaviour can be made of a variety of materials, including polymers [1] and metals [1,2]. This unique behaviour makes auxetic materials very useful in applications where high shear to bulk moduli are required and where densification upon impact is essential, such as in armours and energy absorbing materials.

A variety of natural [3–6] and man-made [7,8] structures, which give rise to auxeticity, have been discussed in the literature. A well known two-dimensional structures are honeycombs with inverted cells [9] and periodic structures with especially designed basic elements. However, while ordered structures are convenient for the analysis of macroscopic deformations in response to external forces, there is currently very little understanding on the modelling of deformations in disordered structures.

This paper has several purposes: (i) to distinguish between two classes of auxetic structures—iso-auxetic and elasto-auxetic; (ii) to demonstrate that elasticity is not fundamental to auxeticity, suggesting that negative Poisson's ratio has limited usefulness as a descriptor of auxetic behaviour; (iii) to write the auxetic strain in terms of local rotational and expansive fields; (iv) to demonstrate that the key role of rotations can be accounted for within the context of symmetric stresses, obviating descriptions using Cosserat theory.

The rationale and the structure of the paper are as follows. In the first part it is demonstrated that auxetic structures may also be isostatic. The term isostatic refers to structures that are statically determinate in mechanical equilibrium. This class of structures is termed here iso-

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auxetic, to be distinguished from more conventional auxetic structures that are termed here elaso-auxetic. In isostatic structures in mechanical equilibrium the internal forces on the scale of the basic elements can be determined uniquely only from balance conditions of forces and of torque moments. This means that in such materials the macroscopic continuous stress tensor is governed by equations that differ significantly from those of elasticity theory, which rely on constitutive stress-strain relations. The isostatic field equations are given below explicitly for two-dimensional systems. The independence of the stress field of compliance-based information obviates descriptions in terms of elastic constants. Since the mechanism for auxetic strain is independent of whether the medium is isostatic or not it then follows that elasticity theory is not fundamental to the modelling of auxeticity in general. This conclusion is explored here. It further suggests that negative Poisson's ratio, a quantity often used to characterise auxetic behaviour, must be understood only as a ratio of strains in perpendicular directions—not as a ratio of elastic moduli.

The existence of iso-auxetic materials has another consequence; on length-scales larger than that of the elements, isostaticity theory gives rise to symmetric stress tensors. This casts doubt on the utility of Cosserat theory, which allows for anti-symmetric stresses and local residual torques for the description of macroscopic auxetic behaviour. The reconciliation of symmetric stress tensors with the significant role that local rotations clearly play in the modelling of auxetic behaviour is done in a following discussion.

The next part of the paper describes a model for strains in auxetic systems. I propose a new expression for the strain in terms of local rotational and expansive fields. The equation is valid for all auxetic materials. The stresses that drive these local fields, however, differ between iso-auxetic and elasto-auxetic materials and it is conjectured that this difference may give rise to significantly different macroscopic behaviour.

2. Isostatic systems, iso-auxetic structures and implications

Consider an auxetic structure made of elements that connect to nearest neighbours at exactly three contact point.[†] Examples of such elements are stars, unfolding elements, and rigid elements, as sketched in figure 1. No translational order is presumed in the structures considered in this paper—the elements are permitted to be irregular in size, shape and orientation across the system. Moreover, a mixture of such elements in one system would work equally well.

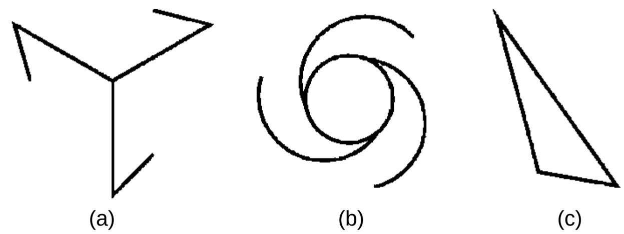

Figure 1. Examples of triangular building blocks: (a) a three-armed star, (b) a folding element and (c) a rigid triangle. Any combination of such elements has the potential to form an iso-auxetic structure.

Consider now an equivalent framework of triangles that represent the basic elements. The edges of the triangles are defined as the lines that extend between the three points of contact that belong to the same element. In structures made of rigid triangles it is straightforward to identify points of contacts, but in some systems these points are not uniquely defined (see, e.g. figure 2). Fortunately, such uniqueness is not essential for the purpose of the following analysis. An example of a particular triangular construction is demonstrated in figure 2 for a structure made of three-armed stars.

The triangles form a connected framework that spans the plane and enclose polygons which we shall term cells.[‡] It will be assumed below that all the cells have even number of triangles around them. This assumption simplifies the discussion in that it ensures that the system possesses a *staggered order* everywhere, a concept that will be discussed in somewhat more detail later. It is possible to lift this assumption using ideas from stress analysis of open-cell foams [10–12], but this is left to a later report.

As long as the contacts between elements can support torque moments there is a finite threshold of loading below which an equilibrium state can be maintained. Suppose then that the structure is in mechanical equilibrium under a given load. At this state the triangles transmit forces through their contacts and these forces are all balanced. The forces exchanged between two neighbouring triangles v and v' are equal and opposite, $\boldsymbol{f}_{vv'} = -\boldsymbol{f}_{v'v}$. Suppose that we wish to solve for the force field. For a structure of N triangular elements there are $3N/2$ contact points,[¶] each of which corresponding to one force vector. Therefore, with two components per force, this gives $3N$ unknowns to solve for.

There are three balance conditions on each triangle; two of force and one of torque moment. It follows that the N triangles give altogether $3N$ equations, exactly matching the number of unknowns. This leads to the conclusion that iso-auxetic systems are statically determinate, or *isostatic*. Since the forces between elements can be solved from balance conditions alone then they are independent of the compliance of the material. Now, the macroscopic stress field is only a coarse-grained representation of these basic

[†]In fact, the following discussion holds for more general systems where the *mean number of contacts* per unit is three.
[‡]The terms and the notations are intended to make contact with a related analysis in solid open-cell foams, where vertices and cells are key concepts [11].
[¶]Effects of the boundary are ignored because for $N \gg 1$ these effects are of order $1/\sqrt{N}$. These effects can be included for finite systems without loss off generality.

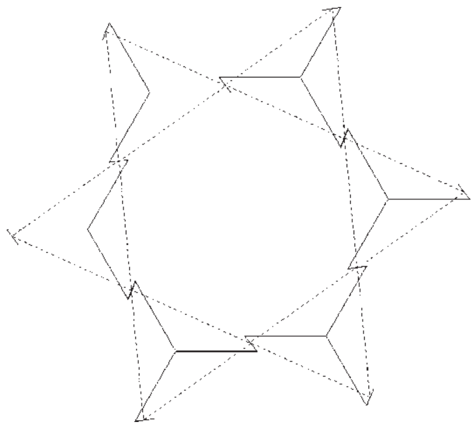

Figure 2. The construction of triangle (dotted lines) for a structure of three-armed stars. The contact points are defined as the midpoints of the short arms at the periphery of the stars. A collection of such elements would give rise to an iso-auxetic structure. Although the figure depicts an ordered structure, this definition applies to disordered structures equally well.

discrete forces. It follows that the continuous stress can also be determined without any information about stress–strain relations.

This is a significant observation—it means that elasticity theory, which resorts to such relations, is redundant for the determination of stresses in iso-auxetic systems. This seemingly surprising conclusion goes beyond isoauxeticity—it holds for most open-cell cellular solids both in two and in three dimensions [11]. This realization has led to the development of a new theory for the stresses that develop in isostatic cellular solids [10–12]. A review of isostaticity theory is outside the scope of this discussion, but it is useful to recall several of its features. At its basis are the following field equations that it give rise to

$$\partial_i \sigma_{ij} = g_j$$

$$\sigma_{ij} = \sigma_{ji} \qquad (1)$$

$$Q_{ij}\sigma_{ij} = 0.$$

In these equations ∂_i is a partial derivative with respect to the coordinate x_i, σ_{ij} is the ij-th component of the stress tensor, g_j is the j-th component of an external force field that may be position dependent (body forces are ignored without loss of generality), and Q_{ij} is a geometric (also termed fabric) tensor that characterises the local microstructure. The equations contain no reference to elastic constants and indeed to any compliance-based information; the only constitutive information is encapsulated in the geometric tensor.

Equations (1) lead to three significant conclusions.

I. It can be shown (see below) that the equations are *hyperbolic*, in stark contrast to the elliptic equations of elasticity theory. This difference is crucial in that, unlike in elastic media, the stress fields that develop in isostatic materials may be strongly non-uniform and are extremely sensitive to the details of the boundary loading [12]. The difference in the stress solutions between iso- and elasto-auxetic materials is bound to give rise to different strain fields and therefore to different macroscopic behaviour.

II. Auxetic behaviour arises from the folding and unfolding of basic elements and as such it is independent of whether the structure is isostatic or not. It is therefore expected that the modelling of auxeticity should be the same for both types of media. Yet, if stresses in isostatic materials are independent of bulk elastic constants then so should be the description of auxeticity in more conventional structures, i.e. in structures with higher mean number of contacts. Moreover, one expects the same description to apply also to structures made of infinitely rigid elements that are non-isostatic, such as the systems of rectangles discussed in Ref. [13]. The redundancy of the elastic constants for modelling auxetic behaviour suggests that it is misguided to interpret Poisson's ratio in auxetic materials as a ratio of elastic moduli. Rather, when using negative Poisson's ratio it must be borne in mind that it can only relate to the ratio of perpendicular strains.

III. As can be seen from the second equation in (1), the stress tensor of isostaticity theory is *symmetric*. This means that residual torque moments vanish on macroscopic length-scales, which can be readily shown from first-principles in any medium that is not coupled to an external torque field. It must be emphasised at this stage that there is no contradiction between the symmetric nature of the macroscopic stress tensor and the fact that rotation of elements is at the core of auxetic behaviour. The reconciliation of the two will be discussed in the next section. This obviates attempts to model large-scale auxetic behaviour using Cosserat theory [14], a theory that allows anti-symmetric components in the stress field.

3. Strains in iso-auxetic structures

Deformations in auxetic materials are intimately linked to geometric structural details. Therefore, the first step towards a basic theory is a method of characterization of the geometry of the structure. This method has to be general in that it can describe any arbitrary disordered structure and it must be useful for the modelling of strains. The geometric (or fabric) tensor Q_{ij} of equation (1) provides such a descriptor.

Consider an iso-auxetic structure, part of which is sketched in figure 3. Make every triangle edge into a vector

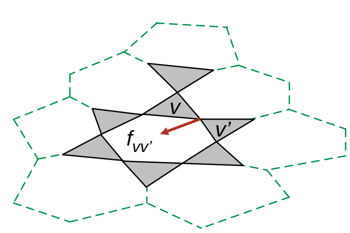

Figure 3. A part of an iso-auxetic structure made of elements that connect to nearest neighbours at exactly three points. The connections make triangles in the plane that partition it into polygonal cells. Four triangular basic elements are shown shaded.

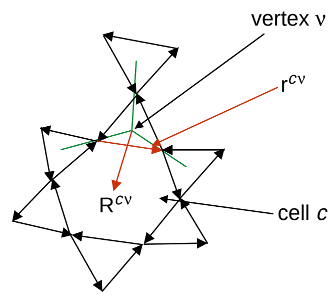

Figure 4. The construction of the geometric tensor Q_{ij} for a disordered iso-auxetic structure. First, the triangle edges are made into vectors $\vec{r}^{\,cv}$ by assigning them directions such that they circulate the triangles in the anticlockwise direction. Vectors $\vec{R}^{\,cv}$ are then defined, which extend from the centroid of triangle v to the centroid of its neighbour cell c. The geometric tensor Q_{ij} is defined as a sum of the outer product of the pairs $r_i^{cv}R_j^{cv}$ over the cells around triangle v, combined with the straightforward rotations described in equation (2).

by assigning it a direction such that the directed edges 'circulate' around the triangle in the anticlockwise direction (see figure 4). Define the centroid of every triangle as the mean position vector of its three corners. Similarly define the centroid of every cell polygon as the mean position vector of the polygon corners. Note that every edge can be indexed uniquely by the triangle v and the cell c that it borders, $\vec{r}^{\,cv}$. Extend now a vector from the centroid of triangle v to the centroid of one of its neighbouring cell $c, \vec{R}^{\,cv}$ (see figure 4).

The tensor Q_{ij} of equation (1) is the symmetric part of the outer product of these two vectors, summed over the three cells around triangle v and transformed as follows

$$Q_{ij}^{v} = \frac{1}{2}\varepsilon_{ik}^{-1}\sum_{c\in v}(r_k^{cv}R_l^{cv} + R_k^{cv}r_l^{cv})\varepsilon_{lj}, \qquad (2)$$

where $\vec{\varepsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the $\pi/2$ rotation in the plane. This tensor turns out to be a natural descriptor of skeletal cellular structures. In addition to playing a key role in the equation that couples the local stress to the local structure, the tensor Q_{ij} is also central to the description of strains in auxetic structures, as we shall see below.

Let the iso-auxetic structure be in mechanical equilibrium under a given external load. Upon increase of the load the local stresses change and the medium deforms. The macroscopically observable strain is the consequence of "constructive superposition" of deformation of triangles. The aim of the following discussion is to express these contributions to the strain field in terms of the structural characteristics and the local stress. The key point is that in the quasi-static limit a change in the local stress would give rise to local rotations of triangles and this, in turn, would add up to effect a global strain. Limiting the discussion to weak external loadings, the global strain has two contributions from local strains—one from rotation of triangles and one from triangle expansion (shrinking can be regarded as an inverse expansion)

$$e_{ij} = e_{ij}^{\text{rot}} + e_{ij}^{\text{exp}}. \qquad (3)$$

It is straightforward to see that in auxetic materials that consist only of ideally rigid triangles (e.g. such as in Ref. [13]) the strain can be described by the rotational term alone. When triangle v rotates by a small angle θ, it contributes to the strain

$$e_{ij}^{\text{rot}} = Q_{ij}\theta. \qquad (4)$$

This expression is identical to the rotational term that gives rise to dilatancy in granular media as it starts to yield [15].[†] The rotational field is position dependent, $\theta = \theta(r)$, and depends on the local stress $\sigma_{ij}(r)$.

Suppose now that the triangles are made of elements that are not rigid but rather can fan out or fold in under local stresses. Then, as the local stress changes, rotation takes place, which also effects expansion of the triangle. The expansion need not be isotropic, namely, the element may expand at different rates in different directions. It is difficult to imagine that in generally disordered structures expansion would not be accompanied by local rotations. The strain due to expansion can be described by

$$e_{ij}^{\text{exp}} = E_{ijkl}\sigma_{kl}, \qquad (5)$$

where, if we confine ourselves to symmetric strains, then the symmetries of the expansion tensor E_{ijkl} are similar to those of the compliance tensor that relates conventional stresses to conventional strains.

Now, since a local rotation is the result of the local stress acting on the triangle then the rotation angle can be expressed in terms of it, $\theta(r) = \theta(\{\sigma_{ij}(r)\})$. Therefore, using this form in equation (4) and upon substitution of relations (4) and (5) into (3), we end up with a stress–strain relation for the iso-auxetic structure

$$e_{ij} = Q_{ij}\theta(\{\sigma_{kl}\}) + E_{ijkl}\sigma_{kl}. \qquad (6)$$

The coarse-graining of this relation presents a challenge. The difficulty is that there are inherent statistical anti-correlations between values of the tensor Q_{ij} on neighbouring triangles [11]. This feature gives rise to fluctuations of the Q_{ij}'s across the system around a vanishing volume average. A detailed discussion of the coarse-graining issue is outside the scope of this presentation, but several points should be emphasised. First, a procedure that overcomes this difficulty for general structures (i.e. for structures that do not possess staggered order) has been developed [12]. The assumption made above that every polygonal cell has an even number of triangles around it leads naturally to a staggered order—one can label triangles by positive and negative such that every positive triangle is surrounded by negative triangles and vice versa. From this one can observe that the rotation angles θ are also

[†]Dilatancy is not mentioned explicitly in Ref. [15], but this phenomenon arises from grain rotations exactly as auxeticity originates in element rotations in auxetic materials. Thus, equation (4) describes both auxeticity and dilatancy in systems of rotating rigid elements. Equation (4) is paralleled by the second term on the right hand side of equation (5.1) in Ref. [15]. The equivalence can be observed by noting that: (i) their relation is for the strain rate rather than strain, i.e. ω_{kl} is the time derivative of my θ, (ii) their p is my Q, and (iii) their relation is given for three-dimensional systems while mine is for two, $p_{ijkl} \Rightarrow Q_{ij}$.

anti-correlated between neighbouring triangles, with all the positive triangles rotating in one direction and all the negative triangles in the opposite. Now, since both the tensor $\vec{Q}$ and the angle θ are anti-correlated between neighbouring grains then their product in equation (6) *must have the same sign across the structure*. This leads to the conclusion that the rotational term must have a well-defined volume average and therefore that it must give rise to a macroscopic observable field. Whether the macroscopic mean corresponds to auxetic or conventional behaviour depends on the geometric state of the structure. The extension of this discussion to "frustrated" systems, containing cells bounded by odd numbers of triangles, is possible and it has been discussed elsewhere in the context of the coarse-graining of the tensor Q_{ij} [12]. The second term of equation (6) suffers from no such complications and its average over the system has a straightforward macroscopically observable consequences. The two terms of equation (6) may interfere "constructively" or "destructively".

It is important to note that the definition of the fabric tensor $\vec{Q}$ is equally valid for elasto-auxetic structures. Moreover, the rotation and expansion of elements in *all* auxetic structures can be described by equation (6).

4. Conclusions and discussion

To conclude, a new understanding has been proposed for the strain that develops in auxetic materials in response to applied stresses. I have shown that isostatic structures can give rise to auxetic behaviour and I have termed this class of materials iso-auxetic. I have demonstrated that stresses in auxetic materials need not necessarily be described by elasticity theory. The reason is that in mechanical equilibrium iso-auxetic materials are governed by isostaticity theory, rather than elasticity theory, and as such these equations do not resort to conventional stress–strain relations.

This conclusion has several implications. One is that it limits the utility of negative Poisson's ratio as a means to obtain strain-stress relations in auxetic materials in general. The ultimate reason for this is that the elastic properties of the static material have very little to do with the manner in which an auxetic material strains.

Another, more significant implication, is that the strain that develops in auxetic materials in general is only a function of the local expansion and rotation of the basic building blocks of the structure. The explicit dependence of the strain on local expansive and rotational fields is given, equation (6). This relation is common to *all* auxetic materials, whether iso-auxetic or not.

It is important to note, however, that equation (6) does not ensure auxeticity. What is offered in this paper is a general formalism to describe strains in systems of rotating and expanding/shrinking connected elements. The formalism is valid independently of *whether the system is auxetic or not*.

To experimentalists and technologists a key question concerns the practical differences between iso-auxetic and elasto-auxetic structures. After all, not only does the strain given in equations (3)–(5) depend on the local rotational and expansive fields in the same way in both classes of materials, but also the dependence of these fields on the local stress, through $\theta = \theta(\{\sigma_{ij}\})$ and equation (5), cannot be affected by the local number of contacts between elements. Moreover, under quasi-static deformations the local expansion and rotational fields can be expressed at all times in terms of the local stress, which can be used to give an explicit relation between the stress and the strain, as indicated in equation (6). It follows that this relation is also universal to all auxetic materials. The real difference between iso-auxetic and elasto-auxetic systems stems from the equations that govern the stress field itself at the beginning of, during, and the end of the deformation process. The stress field equations give rise to different solutions in the two classes of materials and this difference translates into significantly different spatial stress distributions. Specifically, being hyperbolic, equation (2) give rise to non-uniform stress fields, in stark contrast to the uniform solutions of the elliptic field equations of elasticity theory. This difference must result in radically different spatial distributions of rotational and expansion fields in disordered structures, which in turn should lead to distinctly different strain fields. This author believes that these differences should be macroscopically observable.

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