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Two-dimensional Laplacian growth can be mapped onto Hamiltonian dynamics

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Abstract

It is shown that the dynamics of the growth of a two-dimensional surface in a Laplacian field can be mapped onto Hamiltonian dynamics. The mapping is carried out in two stages: first the surface is conformally mapped onto the unit circle, generating a set of singularities. Then the dynamics of these singularities are transformed to Hamiltonian action–angle variables. An explicit condition is given for the existence of the transformation. This formalism is illustrated by solving explicitly for a particular case where the result is a separable and integrable Hamiltonian. This demonstrates that, at least for a family of arbitrary initial conditions, Laplacian growth is an integrable problem.

Much effort has been directed in recent years towards understanding Laplacian growths, both due to the rich variety of patterns that they display and because of their occurrence in many natural and man-made systems. Paradigmatic cases are diffusion-limited aggregation, solidification of supercooled liquid and electrodeposition (for a review see, e.g., Ref. [1]). In spite of more than a decade of intensive research on the problem there is still no theory that can predict the statistics of the patterns that such processes lead to. These moving-boundary problems are deceptively easy to formulate yet very difficult to solve analytically. It has been proposed [2,3] to conformally map the physical surface of such a growth in two dimensions onto the unit circle (UC) and study the evolution of the singularities of the map [4]. The resulting equations of motion (EOMs) are strongly coupled nonlinear first order ODEs that are difficult to solve, other than for special cases [5]. Another difficulty with this approach is that the formalism breaks down after a finite time because singularities of the conformal map travel to the UC and eventually hit it, at which time the map ceases to be analytic. This breakdown is manifested in cusp singularities that form on the physical surface in the absence of surface tension [6]. A very important question in this approach is whether the system is integrable or even Hamiltonian. It has been found that the problem enjoys a set of conserved quantities [3,7], but it is unclear how these quantities can assist in finding an energy-like functional in the problem.

In this paper it is shown that it is possible to transform the dynamics that govern the growth of the surface into Hamiltonian dynamics as long as the EOMs hold (namely, up to the cusp formation). This is done in two steps: First the EOMs of the singularities of a general map are written down and then their space coordinates are transformed into action–angle variables. The equations for this transformation are given

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for any general initial surface. Although the general existence of a solution for these equations is not rigorously proven here, I show that the surface evolves according to Hamilton's equations, and an example is given where a particular system is solved exactly and is shown to be integrable for a family of arbitrary initial conditions. The action variables are constants of the motion and an energy functional can be clearly identified. The importance of such a mapping cannot be overemphasized: (i) By producing such a transformation the system is demonstrated to be Hamiltonian and possibly also integrable, which is an important issue in itself; (ii) A large body of knowledge that has been accumulated through many decades of studying Hamiltonian systems is consequently made accessible to such growth problems. This can immediately pave the way to a theory of Laplacian growth, as will be reported elsewhere [8].

I start by formulating the problem and briefly presenting the EOMs of the singularities that are generated by the conformal map of the surface onto the UC. Consider a simply connected line surface, $\gamma(s)$, embedded in two dimensions which is parametrized by $0\leqslant s<2\pi$, and which is fixed at a given electrostatic potential. A higher potential is assigned to a circular boundary whose radius tends to infinity. The potential field, Φ, outside the area enclosed by γ is determined by Laplace's equation

$$\nabla^2\Phi=0\,. \tag{1}$$

The surface is assumed to grow at a rate that is proportional to the local electrostatic field normal to the surface

$$v_n=-\nabla\Phi\cdot\hat{\boldsymbol{n}}\,.$$

Shraiman and Bensimon [4] have shown that the surface evolves in time, t, according to

$$\partial_t\gamma(s,t)=-\mathrm{i}\partial_s\gamma(s,t)\times\{|\partial_s\gamma(s,t)|^{-2}+\mathrm{i}g[|\partial_s\gamma(s,t)|^{-2}]\}\,, \tag{2}$$

where g is a real function that corresponds to a physically insignificant "slide" of a point along the surface. Without that last term the EOM can be written as

$$\partial_t\gamma(s,t)=-\mathrm{i}\frac{\delta s}{\delta\gamma^*}\,, \tag{3}$$

where the asterisk stands for complex conjugate. Eq. (2) is the limit of a conformal map $\zeta=F(z)$ that maps the UC onto the physical surface in the ζ-plane through $\gamma(s,t)=\lim_{z\to e^{is}}F(z,t)$. The map considered here is a general ratio of two polynomials of the same degree. This requirement results from the need that the topology far away from the growth remains unchanged, so as to retain the original boundary conditions on a circle far away and the generic logarithmic divergence of Φ when $z\to\infty$ [9].

$$F'\equiv\frac{\mathrm{d}F(z)}{\mathrm{d}z}=\prod_{n=1}^{N}\frac{z-Z_n}{z-P_n}\,, \tag{4}$$

where $\{Z\}$ and $\{P\}$ are the zeros and the poles of the map, respectively. It can be shown [4,9] that the dynamics of these singularities are governed by the EOMs

$$-\dot{Z}_n=Z_n\left(G_0+\sum_{m'}\frac{Q_n+Q_{m'}}{Z_n-Z_{m'}}\right)+Q_n\left(1-\sum_m\frac{Z_n}{Z_n-P_m}\right)\equiv f_n(\{Z\};\{P\})\,,$$

$$-\dot{P}_n=P_n\left(G_0+\sum_m\frac{Q_m}{P_n-Z_m}\right)\equiv g_n(\{Z\};\{P\})\,, \tag{5}$$

where

$$Q_n=2\prod_{m=1}^{N}\frac{(1/Z_n-P_m^*)(Z_n-P_m)}{(1/Z_n-Z_m^*)(Z_n-Z_{m'})}\,,$$

$$G_0=\sum_{m=1}^{N}\frac{Q_m}{2Z_m}+\prod_{m=1}^{N}\frac{P_m}{Z_m}\,,$$

and where the primed index indicates $m'\neq n$.

We now wish to introduce a new set of canonical coordinates that can transform this system of coupled first order ODEs into a Hamiltonian system. There are two reasons why one should expect to find a Hamiltonian at all: The first stems from the existence of constants of the motion, as found by Richardson [3]. The second reason relates to the fact that already the EOM of the surface can be written in the form of Hamilton's equations as follows: Eq. (2) is the limit of an EOM for the map F as $z\to e^{is}$:

$$\frac{\partial F'}{\partial t}=\frac{\partial}{\partial z}(zF'G)\,, \tag{6}$$

where G is the analytic function whose limit is the braces on the r.h.s. of (2). Using the identity

$$\frac{\partial F'}{\partial t}\frac{\partial t}{\partial z}\frac{\partial z}{\partial F'}=-1$$

we obtain that

$$\frac{\partial z}{\partial t}=-\frac{\partial(zF'G)}{\partial F'}\,. \tag{7}$$

Eqs. (6) and (7) can then be interpreted as Hamilton's relations if z and F' are interpreted as canonical field variables and $zF'G$ as the "Hamiltonian density". Alternatively, Eq. (3) immediately shows the underlying Hamiltonian structure of the EOM.

Since the dynamical system (5) represents the motion of the curve it is plausible that this system can also be derived from a Hamiltonian. To this end let us start from the desired result and work our way backwards. The goal is to obtain a Hamiltonian of separable variables (although separability is not necessarily a constraint)

$$H=H(\{\boldsymbol{J}\};\{\boldsymbol{\Theta}\})\,,$$
$$\boldsymbol{J}\equiv(J_1,J_2,\ldots,J_N),\quad \boldsymbol{\Theta}\equiv(\Theta_1,\Theta_2,\ldots,\Theta_N)\,. \tag{8}$$

If this system is integrable then $H=H\{\boldsymbol{J}\}$ and we have a motion on a N-dimensional torus. At this stage it is possible to choose arbitrarily the form of the target Hamiltonian into which the system should be mapped, but for simplicity we require here that the Hamiltonian has the form

$$H=\sum_{n=1}^{N}\omega_n J_n\,. \tag{9}$$

It should be emphasized though that the formulation presented here is in no way restricted to this form. These coordinates are required to obey Hamilton–Jacobi EOMs,

$$\dot{\Theta}_n=\frac{\partial H}{\partial J_n},\quad \dot{J}_n=-\frac{\partial H}{\partial \Theta_n}\,. \tag{10}$$

The transformation we seek should yield $J_n=J_n(\{Z\};\{P\})$ and $\Theta_n=\Theta_n(\{Z\};\{P\})$. Combining Eqs. (5), (9) and (10) we obtain that to effect the desired transformation the following set of equations needs to be satisfied,

$$\dot{J}_n=\sum_{m=1}^{N}\left(\frac{\partial J_n}{\partial Z_m}f_m+\frac{\partial J_n}{\partial P_m}g_m\right)=-\frac{\partial H}{\partial \Theta_n}=0\,, \tag{11}$$

$$\dot{\Theta}_n=\sum_{m=1}^{N}\left(\frac{\partial \Theta_n}{\partial Z_m}f_m+\frac{\partial \Theta_n}{\partial P_m}g_m\right)=\frac{\partial H}{\partial J_n}=\omega_n\,, \tag{12}$$

where the r.h.s. of these equations is particular to the Hamiltonian that we have chosen in Eq. (9). This set of equations can be written generally for any system in the form

$$[\boldsymbol{f}(\boldsymbol{\Gamma})\cdot\nabla]\boldsymbol{J}(\boldsymbol{\Gamma})=\boldsymbol{\omega}\,. \tag{13}$$

In this notation $\boldsymbol{f}$, $\boldsymbol{J}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\omega}$ are $2N$-component vectors,

$$\boldsymbol{f}\equiv(f_1,f_2,\ldots,f_N,g_1,\ldots,g_N)\,,$$
$$\boldsymbol{\Gamma}\equiv(Z_1,Z_2,\ldots,Z_N,P_1,\ldots,P_N)$$
$$\boldsymbol{J}\equiv(J_1,J_2,\ldots,J_N,\Theta_1,\ldots,\Theta_N)\,,$$
$$\boldsymbol{\omega}\equiv\left(-\frac{\partial H}{\partial\Theta_1},-\frac{\partial H}{\partial\Theta_2},\ldots,-\frac{\partial H}{\partial\Theta_N},\frac{\partial H}{\partial J_1},\ldots,\frac{\partial H}{\partial J_N}\right),$$

and ∇ is the gradient in the $2N$-dimensional space of $\boldsymbol{\Gamma}$. If $\boldsymbol{\omega}$ is a vector of N zeros and N constants of the motion, as chosen in Eq. (9), then (13) is a linear set and therefore has a solution as long as the operator $\boldsymbol{f}(\boldsymbol{\Gamma})\cdot\nabla$ has no vanishing eigenvalue. Thus, unless such a singular case occurs, this set of equations does have a solution for $\{\boldsymbol{J}(\boldsymbol{\Gamma})\}$ and $\{\boldsymbol{\Theta}(\boldsymbol{\Gamma})\}$. This solution, when it exists, defines a specific transformation from the chosen Hamiltonian to the problem of the dynamics of the singularities, and hence to the original growth problem. The existence of such a solution immediately points to the integrability of the system.

Having discussed the general case, it is useful to consider a specific example of a class of initial conditions where the solution to the set of equations (11) and (12) indeed exists and can be obtained explicitly. Assume that at $t=0$ the initial surface can be represented by

$$\gamma(s,0)=e^{is}-\sum_{j=1}^{2}R_j\ln[e^{is}-P_j(0)]\,, \tag{14}$$

where

$$R_j=(-1)^j\{[P_j^2(0)-Z_1^2(0)]/[P_1(0)-P_2(0)]\}$$

and $|P_1|$, $|P_2|$, $|Z_1|<1$. The surface $\gamma(s,t)$ can be shown to retain this form for any later time by substituting for P_j and Z_j their time-dependent values. This form is valid for any number of singularities when $-R_j$ is replaced by the residues of the analytic map F. The growth problem consists now of finding the dynamics of two zeros at $Z_1(t)$ and $Z_2(t)=-Z_1(t)$ and two poles at $P_1(t)$ and $P_2(t)$. The trajectories of these singularities can be found by substituting into (5)

$$-\dot{Z}_1=Z_1\left(\frac{2Q_1}{Z_1}-\frac{Q_1}{Z_1-P_1}-\frac{Q_1}{Z_1-P_2}-\frac{P_1P_2}{Z_1^2}\right),$$

$$-\dot{P}_j=P_j\left(\frac{Q_1}{Z_1}+\frac{Q_1}{P_j-Z_1}-\frac{Q_1}{P_j+Z_1}-\frac{P_1P_2}{Z_1^2}\right). \quad (15)$$

It can be shown [9] that for the map to be analytic this 2×2 system has to obey the relation $Z_1+Z_2=P_1+P_2$. This immediately simplifies the EOMs to the form

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_1^2=2P_1P_2(1-2K),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_j=\frac{P_j}{Z_1^2}\left[P_1P_2+(P_j^2+Z_1^2)K\right], \quad (16)$$

where $K\equiv(1-Z_1P_1^*)(1-Z_1P_2^*)/(1-|Z_1|^4)$. I choose the initial conditions such that $|P_j(0)|$ is smaller than $|Z_1(0)|$ and also $\arg(|P_j|)=\arg(|Z_j|)$. This choice is inconsequential to the exact solution below and for the purpose of the present discussion. For the opposite choice, $|P_j(0)|>|Z_1(0)|$, the growth process is stable (namely, no cusp will form) and the solution holds to $t\to\infty$. The unstable case is chosen to emphasize the thrust of this paper that the system is integrable until the EOMs loose their validity regardless of whether this occurs at a finite or infinite time.

A rather tedious manipulation of Eqs. (16) yields that the following is a constant of the motion,

$$J_j=\frac{1}{Z_j^2-P_j^2}+\zeta_j=\text{const}, \quad (17)$$

where $\zeta_j=\frac{1}{8}\ln[(1+P_j^2)/(1-P_j^2)]$. These constants we can immediately identify as the action variables. By substituting for Z_1^2 from (17) and employing some algebra the solution for P_j^2 is found,

$$t=\tfrac{1}{2}\ln[P_j^2(J_j-\zeta_j)^8]-\int^{\zeta_j}\frac{2\,\mathrm{d}\zeta}{\tanh 4\zeta(J_j-\zeta)^2}. \quad (18)$$

Using relations (17) and (18) we obtain the solution for Z_1 in a similar form. From the EOMs it can be verified that from the above initial conditions the zeros propagate towards the UC faster than the poles, and hence the solution exists as long as the singularities are within the UC. Several stages of the growth are shown in Fig. 1, where the surface evolves from mildly oval into an eye-shaped structure. In the mathematical plane the evolution consists of the poles and the zeros moving radially towards the UC.

It should be noted that the above treatment assumes a unity factor in front of the map F. In fact there should also appear a purely time-dependent prefactor which takes care of the total area of the growth increasing linearly with time (the assumption is that the flux into the growth is constant in time). By setting this prefactor to unity the surface is effectively rescaled at each time step, which is why in Fig. 1 sections of the boundary seem to retreat with time. The evolution of this prefactor can be very simply incorporated into the formulation, but for clarity it has been omitted here.

Since we have now solutions in the form of Eq. (18),

$$t=I_{\Gamma_j},\quad \Gamma\equiv(Z_1,Z_2,\ldots,Z_N,P_1,\ldots,P_N),$$

we can use Eqs. (11) and (12) to find the action-angle variables in terms of the original coordinates. From the integrals of motion we obtain

$$\begin{pmatrix}J_k\\ \Theta_k\end{pmatrix}=\sum_{j=1}^{2}\begin{pmatrix}a_{jk}\\ c_{jk}\end{pmatrix}I_{Z_j}+\begin{pmatrix}b_{jk}\\ d_{jk}\end{pmatrix}I_{P_j}, \quad (19)$$

where a, b, c, and d are constants. These coefficients of the action–angle variables are required to obey

$$\sum_{j=1}^{2}\begin{pmatrix}a_{jk}+b_{jk}\\ c_{jk}+d_{jk}\end{pmatrix}=\begin{pmatrix}0\\ \omega_j\end{pmatrix}.$$

It may seem from this result that there are too many coefficients that can be chosen arbitrarily. But for a general growth these coefficients are complex and the requirement that the action and angle variables take on only real values reduces the arbitrariness to exactly four independent coefficients with possible other four arbitrary integers (corresponding to multiplicity of $2\pi i$ when setting the imaginary parts of the variables to zero). The angle variables, Θ_k, thus grow linearly with time while the action variables, J_k, are constants of the motion, and by transformation (19) we have obtained the target Hamiltonian (9).

To conclude, I have shown here that the growth of

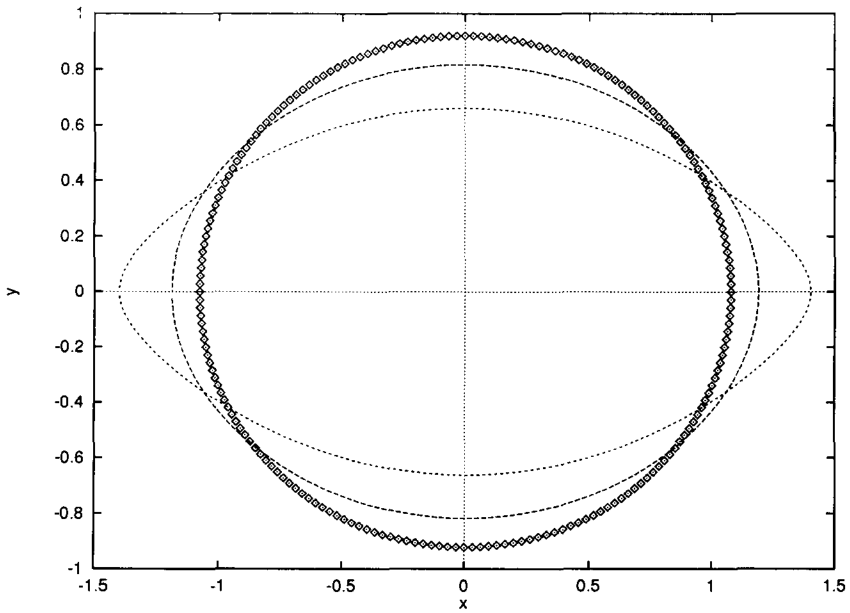

Fig. 1. The growth of the surface discussed in the text for two zeros $Z_j^2(t_0)=0.55$ and two poles $P_j^2(t_0)=0.005$.

a free surface in a Laplacian field is governed by Hamiltonian dynamics which is integrable if (13) has a solution. This formalism holds until cusp singularities occur. I have shown that the surface evolves according to Hamilton's equations if F' and z are interpreted as the canonical field variables and the Hamiltonian density is $zF'G$. A family of initial conditions has been analysed explicitly where integrability can be demonstrated, showing that at least for some classes of arbitrary initial conditions relation (13) does have a solution. I have chosen the simple Hamiltonian given in (9) to illustrate how this transformation can be carried out, and have formulated the transformation equations for a general surface. Several questions still remain: (i) Is there a case where the operator $f(\Gamma)\cdot\nabla$ in Eq. (13) has at least one vanishing eigenvalue? And if so what is the structure it corresponds to? (ii) How constrained are we in choosing the Hamiltonian so that the transformation equations still have a solution? These and other questions are currently under investigation. This author believes that the example given here represents only a "hydrogen model" system where direct mapping to Hamiltonian dynamics is possible, and that such a mapping is a general property of Laplacian growth processes. Therefore this approach can pave the way towards a theory of Laplacian (and possibly more general) growth processes, as will be discussed in a later publication that is currently under preparation.

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References

[1] P. Pelce, Dynamics of curved fronts (Academic Press, New York, 1988);
D.A. Kessler, J. Koplik and H. Levine, Adv. Phys. 37 (1988) 255;
P. Meakin, in: Phase transitions and critical phenomena, Vol. 12, eds. C. Domb and J.L. Lebowitz (Academic Press, New York, 1988);

T. Vicsek, Fractal growth phenomena (World Scientific, Singapore, 1989).
[2] L.A. Galin, Dokl. Akad. Nauk USSR 47 (1945) 246;
P.Ya. Polubarinova-Kochina, Dokl. Akad. Nauk USSR 47 (1945) 254; Prikl. Mat. Mekh. 9 (1945) 79.
[3] S. Richardson, J. Fluid Mech. 56 (1972) 609.
[4] B. Shraiman and D. Bensimon, Phys. Rev. A 30 (1984) 2840.
[5] L. Paterson, J. Fluid Mech. 113 (1981) 513; Phys. Fluids 28 (1985) 26;
S.D. Howison, J. Fluid Mech. 167 (1986) 439;
D. Bensimon and P. Pelce, Phys. Rev. A 33 (1986) 4477;
S. Sarkar and M. Jensen, Phys. Rev. A 35 (1987) 1877;
B. Derrida and V. Hakim, Phys. Rev. A 45 (1992) 8759.
[6] D. Bensimon, L.P. Kadanoff, S. Liang, B.I. Shraiman and C. Tang, Rev. Mod. Phys. 58 (1986) 977;
W.-s. Dai, L.P. Kadanoff and S. Zhou, Phys. Rev. A 43 (1991) 6672.
[7] M.B. Mineev-Weinstein, Physica D 43 (1990) 288;
M.B. Mineev-Weinstein and S.P. Dawson, preprint (1993).
[8] R. Blumenfeld, in preparation.
[9] R. Blumenfeld and R.C. Ball, in preparation.