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NOVEL FLUX SOLUTIONS IN NONLINEAR CONDUCTING CONTINUUM SYSTEMS WITH NEGATIVE DYNAMIC RESISTANCE

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A nonlinear conducting continuum system with a negative dynamic resistance is studied. The metastable solutions to Maxwell's equations are proposed to consist of narrow fluxes of currents and their properties are discussed. In systems containing pinning impurities an array of fluxes appears and a critical external current density is found, above and below which the system's behaviour is governed by the interaction and the dissipation, respectively. The size dependence of this critical value is found to be nontrivial.

1. Introduction

In this paper I discuss nonlinear conducting media whose J–E relation is of the form

$$J(r) = \sigma(r)\,|\,E(r)|^{\gamma-1}E(r)\,. \tag{1}$$

Real systems displaying such a behaviour have been discussed in the literature [1]. This equation, combined with the continuity relation

$$\operatorname{div} J = 0\,, \tag{2}$$

has been shown to suffice for determining a unique solution to the electrostatic Maxwell equations whenever $\gamma > 0$ [2]. For $\gamma < 0$, it was shown in discrete systems that there appear metastable solutions whose number may grow exponentially with the system's size [3]. In that case each metastable solution corresponds to a given configuration of current directions in the individual branches of the network. For the continuum systems, discussed here, there is yet no understanding of the nature of the different solutions. In this paper I concentrate on the range $-1 < \gamma < 0$, and argue that in completely homogeneous such conducting materials, each solution corresponds to a narrow flux through which the entire current flows from one boundary to its opposite. Different solutions then correspond to different locations of this flux. Systems exhibiting such a negative resistance regime are, to a very good approximation, tunnel diodes (within a considerable range of their I–V curve), and also two-dimensional Josephson junctions [4] ($\gamma \approx -1$). I next discuss the case when there are small impurities that can pin a flux locally, so that many current fluxes traverse the medium. I find a different behaviour of the system when the external current density is smaller or larger than some critical value, J_{0c}. This value is found to depend nontrivially on the aspect ratio of the bulk dimensions and to typically vanish when the separation between the terminals, L, becomes very large. When the flux lines are allowed to be very tortuous the dissipation energy is higher than in the straight line state and the critical current density is found not to depend on L when the bulk approaches a two-dimensional geometry.

2. Steady state solution

Consider a system constructed of the above nonlinear material occupying a volume between two conducting plates of surface area R^2 parallel to the x–y plane and located a distance L apart in the z direction, which are held at different potentials. The density of dissipated energy in this medium is given by [2]

$$u(r) = (\gamma + 1)^{-1}\sigma(r)\,|E(r)|^{\gamma+1}\,, \tag{3}$$

which, for $-1 < \gamma < 0$, is minimised locally by reducing the electrostatic field. However, inspecting (1) and (3), one can see that this leads to *maximisation* of the local current density. It follows that it is favourable for the system to drive as much current as possible through an as small as possible section area. This is in sharp contrast to the ordinary case of positive dynamic resistance, $\gamma > 0$, where minimisation is achieved when the current is distributed as thinly as possible. Consequently, in our medium, given that it is homogeneous, the current will collapse into a narrow flux, whose location in the x–y plane is determined by initial conditions and the distribution of local geometrical constraints (like grain boundaries, etc.). The width of this flux is determined by microscopic processes, whose nature is beyond the thrust of this paper.

Next assume that the material contains impurities that do not let the current collapse into one flux. The current will then split to a number of fluxes, each trapped in the potential well of a pinning impurity and one ends up with an ensemble of fluxes extending between the terminal plates. This paper is mainly concerned with the properties of this system. As an illustrative example consider the case of just two such identical fluxes, each of which having radius r and whose centers are separated by a distance a. These can be envisaged as two wires carrying currents i in the same direction. Such two wires attract (see below for a detailed expression) and some mechanism is needed to pin them down, which is provided by the above postulated pinning impurities. For simplicity I assume that: (i) the fluxes are cylinders of radius r_0, all carrying the same current i_0 (ii) the conductivity of the material is constant, $\sigma(r) = \sigma$, on length scales larger than the range of typical pinning potential well, but smaller than the system's size, and (iii) that the density of flux lines per unit area, n, is constant all over the sample. These amount to treating the system by a mean field approach, because under such assumptions each flux experiences the same field. It is evident that the situation is usually more complicated than this. For example, the current amplitudes in the fluxes can be distributed differently, the spatial distribution of the flux locations need not be uniform and their shapes may also be very convoluted. However, even the above naive simplifications give rise to interesting phenomena and can serve as a framework for treating the system.

There are three typical energies that determine the physics of this system: the dissipation within the fluxes, the interaction between fluxes and the interaction between the fluxes and the pinning defects. Not being concerned here with the pinning mechanism, I will consider the fluxes to be practically frozen. However, a flux will eventually be allowed to fluctuate around the frozen straight state (see below). If the external current density is J_0, then the current that each flux carries is $i_0 = J_0/n$ and the energy dissipated in all of them is

$$U_{\mathrm{d}} = VJ_0^{1+1/\gamma}(n\pi r_0^2\sigma)^{-1/\gamma} , \tag{4}$$

where $V = LR^2$ is the volume of the system. To find the interaction energy consider first the case of parallel flux lines. It is a textbook excercise to find that two lines, a distance r apart, experience an attractive force of the form

$$F_2 = \mu i_0^2\{[1 + (L/r)^2]^{1/2} - 1\}/2\pi , \tag{5}$$

where μ is the permeability constant. Integrating over the distance in $F_2(r)$, the interaction energy between a pair is found to be

$$U_2 = \mu i_0^2 L([1 + (r/L)^2]^{1/2} - r/L - \ln\{L/r + [1 + (L/r)^2]^{1/2}\})/2\pi . \tag{6}$$

Under the assumption that the density of fluxes is high, the total interaction in the system is found by choosing a flux, integrating to find its energy of interaction with the entire system, and then summing over all fluxes. This leads to

$$U_{\mathrm{int}} = (\mu J_0^2 V L^2/6) \times \Big([2(R/L)^2 - 1][1 + (R/L)^2]^{1/2} - [2(a/L)^2 - 1][1 + (a/L)^2]^{1/2} - 2(R^3 - a^3)/L^3 - 3(R/L)^2 \ln\{L/R + [1 + (L/R)^2]^{1/2}\} + 3(a/L)^2 \ln\{L/a + [1 + (L/a)^2]^{1/2}\} \Big) , \quad (7)$$

where a is now the separation between two neighbouring fluxes and will, for the moment, be assumed to be much smaller than both R and L. This expression can be simplified in the following limits:

$$U_{\mathrm{int}} \approx \begin{cases} -\mu J_0^2 V R^2 \ln(L/R)/2 , & L \gg R \gg a , \\ [\sqrt{2} - 1 - 3\ln(1+\sqrt{2})]\mu J_0^2 V L^2/6 , & L \approx R \gg a , \\ -\mu J_0^2 V R L/2 , & R \gg L \gg a . \end{cases} \quad (8)$$

The first and second relations in (8) correspond, respectively, to a rod-like and a cubic bulk shape, while the third corresponds to a flat geometry. The dependence of the interaction energy on the size of the bulk differs significantly in the three regimes. Fig. 1 shows the value of the dimensionless quantity $Y \equiv -2U_{\mathrm{int}}/\mu V(J_0 R)^2$ as a function of L/R.

3. The competition between interaction and dissipation

Let us now consider the relative significance of the interaction and dissipation. If the latter is very large, the lines will be essentially straight, unless otherwise dictated by the local grain boundaries, etc. If the reverse is true, the attractive interaction dominates the system's behaviour and may even distort the flux lines (but not unpin any flux line entirely, see discussion below). Since the system is in steady state rather than in equilibrium, there arises the question of how to compare the effect that these two quantities have on the system. Without interaction between the fluxes, one minimises the dissipation for finding the steady state, while without the dissipation the free energy is minimised for finding the equilibrium state. To avoid this fundamentally difficult issue, let us note that the dissipation increases linearly with the length of the flux as it bends and twists across the medium. This suggests that we may consider a spring-like potential energy that is proportional to the dissipation u_{d} [#1], and treat the system as in equilibrium. Therefore we can now compare the two effects by studying the ratio $X = |U_{\mathrm{d}}|/|U_{\mathrm{int}}|$. We can see from (4) and

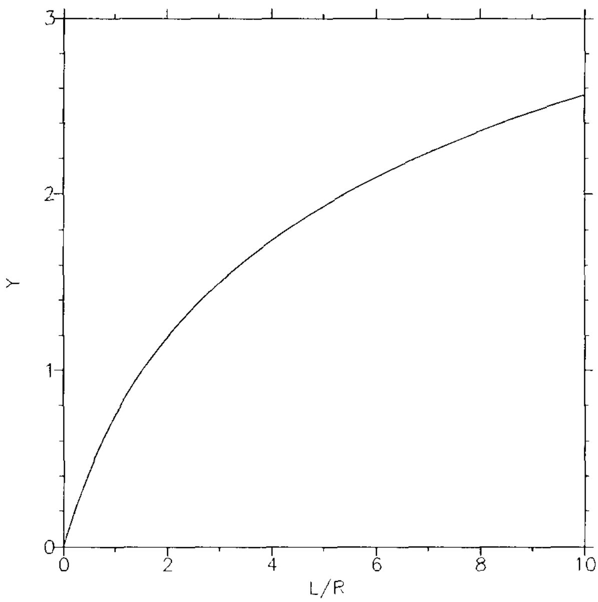

Fig. 1. The dependence of the dimensionless interaction energy $Y \equiv -2U_{int}/V\mu(J_0R)^2$ on the aspect ratio L/R. The value of a/R is taken here to be 0.01.

(8) that, for a given external current density, X always decreases with size, which implies that the interaction dominates the system's behaviour in the thermodynamic limit.

This ratio, however, depends monotonously on J_0. Hence by varying the external current density J_0, one can change X at will. Examining (4) and (8), it follows that there exists a critical value J_{0c} such that for J_0 smaller than J_{0c} the dissipation governs the line shapes and vice versa. This critical value corresponds to some constant value of X, and, therefore, by inverting the ratio above, the critical external current density is found to have the form

$$J_{0c} \sim \begin{cases} [R^2 \ln(L/R)]^{-\delta}, & L \gg R, \\ (RL)^{-\delta}, & R \gg L, \\ L^{-2\delta}, & R \approx L, \end{cases} \tag{9}$$

[#1] A way to find the energy balance is to actually compare the interaction energy gained by reorienting the fluxes in a new state, to the energy dissipated in the process. The latter is the power integrated along the time it takes a typical flux to move from the initial to the final states. Postulating naively that this movement is either adiabatic or at constant small velocity, the results below are similarly obtained.

where $\delta \equiv |\gamma|/(|\gamma|+1)$. Inspecting (9) one can observe that J_{0c} decreases when: (i) L increases at a given R for $L \gg R$, (ii) either R or L increases for $R \gg L$, or when (iii) the system's size increases for $L \approx R$. The rates of this decrease are different and are characteristic to the different regimes.

4. Tortuous fluxes

Let us now relax the above assumptions and allow for fluctuations of the flux lines around the straight state. The mechanism causing these fluctuations is not relevant for the purpose of the following analysis, but some possibilities are discussed below. For simplicity, it will be assumed that the set of distorted fluxes can be related one-to-one to the former set. This enables to use the above discussion without worrying about significant changes to the spatial distribution of fluxes in the medium. When the fluctuations are small, both dissipation and interaction energies retain the above qualitative dependence on the system's dimensions and the above analysis still applies. Let now the fluctuations become wilder. In the extreme case, each flux may assume a random-walk-like shape. Assuming the fluxes do not wind around each other, the functional form of the interaction energy does not change much (up to a possible numerical factor) due to the many changes in the orientation of the current, which cancel out in the calculation of the net force F_2.

The dissipation energy, however, does change. Under the assumption that a is much smaller than both R and L, and approximating each flux as confined by its neighbours to a tube of radius a (to avoid intersection and convergence with neighbouring fluxes, thus following the above assumption that the concentration of fluxes does not undergo significant changes), the dissipation increases proportionally to the increase in the length of the flux. Due to the fact that the random walk fills the tube, the length of the flux is of order aL. Hence the dissipation increases by a factor of a and the above bulk size dependence of X is retained. It follows that the L-dependence of J_{0c} does not change as well. However, keeping R fixed and reducing L, such that $R \gg a \gg L$, the length of a random-walk line becomes of order L^2. The interaction retains the limiting functional dependence on L given in (8) and X can now be written in the form

$$X \approx 2J_0^{1/\gamma-1}(n\pi r_0^2\sigma)^{-1/\gamma}/R\,, \tag{10}$$

which is *independent of* L. It follows that J_{0c} reaches a finite lower value as L decreases towards the two dimensions limit. Note that this result holds only if the flux lines resemble random walks. If their length scales as L^ν with ν strictly smaller than two, then the right-hand side of (10) should be multiplied by $L^{\nu-2}$ and J_{0c} decreases with L, although slower than in the former case of straight lines.

5. Discussion

To conclude, it has been suggested here that the nonlinear conductivity law (1), with $-1 < \gamma < 0$, allows for stable local maxima of the current density, which in the presence of pinning impurities leads to separation of the current into individual fluxes. I have discussed the mechanisms that govern the behaviour of the fluxes and found a critical external current density at which the system crosses over from being determined by the interaction to being controlled by the dissipation. This critical value has been found to decrease with the system's size, as long as either the fluxes do not assume random-walk-like shape, or the system is essentially three dimensional. When the flux lines do perform random walks, there appears a crossover from three to two dimensions, where the critical current density reaches a finite lower value independent of L. Note that in this context, the system becomes two dimensional when $L \ll a$ rather than merely $L \ll R$.

Distortions of the fluxes from the straight line shape can occur due to: (i) local geometrical constraints that force the current fluxes to bend, (ii) high local attractive interactions that are sufficient to unpin parts of flux lines and also to overcome the additional dissipated power, and (iii) thermal fluctuations that are sufficiently large to disturb the straight line state.

One implication of the above analysis is the possible occurrence of a "distortion transition" due to the increase of the interaction-to-dissipation ratio. When this ratio is small, the lengths of the fluxes are dominated by the dissipation and are in principle proportional to L. As it increases, the fluxes may favour highly tortuous configurations to minimise the free energy.

The introduction of temperature fluctuations, as another distorting mechanism, is subject to some limiting constraints, if the analysis presented here is to remain valid. In the foregoing it has been implicitly assumed that the temperature may be high enough to unpin flux lines from the pinning potential wells, but that it is not sufficient to remove them too far away from their initial immediate neighbourhood. Thus, increasing the temperature creates a new set of flux lines that can be topologically mapped in a one-to-one fashion into the set for lower temperature, where the lines are approximately straight. It is also implicitly assumed that the typical time for a flux line to change shape is long compared to the time of observation, so that a frozen system of flux configurations can be considered. It is interesting to study this system without these assumptions. Unlike the physics of magnetic fluxes penetrating superconducting media, the flux–flux interaction in our system is attractive only. Hence, while magnetic fluxes may form a two dimensional lattice in superconducting materials, our current fluxes cannot (unless the pinning defects themselves are prearranged in such a lattice). Nevertheless, the study of these conducting systems may yield insight for the statistics of fluxes in other media, e.g., for current carrying attracting polymers or for reversed miscelles.

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