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Dynamics of fracture propagation in the mesoscale: Theory

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Abstract

A recent theoretical model (Blumenfeld, Phys. Rev. Lett. 76 (1996) 3703) is described for modes I and III crack propagation dynamics in noncrystalline materials on mesoscopic lengthscales. Fracture has been one of the longest standing problems in physics and materials science, and despite much effort, several fundamental issues have stubbornly resisted resolution:

(i) Running cracks reach a steady-state velocity of roughly half the shear wave speed, while theoretical predictions based on energetics are twice as high. The discrepancy originates from dynamics, but a consistent dynamical model has been slow to emerge.

(ii) There is little understanding of the mechanisms for crack initiation and arrest and the hysteresis between them. Lattice trapping, although relevant on the atomic scale, cannot explain this phenomenon on mesoscopic and macroscopic scales.

(iii) Another intriguing phenomenon is appearance of velocity periodic oscillations in some materials and the relation between this and material properties.

(iv) As a result of the above issues, there is currently no consensus on the form of the equations of motion that govern mesoscale fracture dynamics.

Whether explicitly or implicitly, most traditional models use quasi-static and near-equilibrium concepts to analyse the dynamics of propagation. It is argued here that such approaches are bound to fail. Two reasons are responsible for this and consequently for the dire understanding of this problem: First, most fast fracture processes are usually restricted to post-mortem measurements of the already fractured system, while the process itself is too fast to capture. Only recently there emerged experiments where the dynamic process is continuously monitored. Second, it is strongly contended here that the fracture phenomenon is governed by *different mechanisms on different length-scales*, a crucial aspect that has not received sufficient attention. In ideally brittle propagation, the crack is atomically sharp and therefore atomic potentials are important (5–10 Å). Anharmonicity plays a significant role on this scale due to large local strains at the crack tip, which gives rise to a strong nonlinear behaviour. On large scales (>μm), continuum linear elasticity describes quite well the stress field and the far-away elastic energetics. This is exactly because cracks propagate slower than the bulk speed of sound, which allows the bulk stress to relax to its static value in the frame of the moving crack. Ultimately, this is the reason why contour integral calculations of energy influx into the crack tip are valid as long as the contours are taken well away from the tip. Between the atomic and the continuous scales there are at least two more relevant length-scales: One is that of the cohesive zone, which is the region where the continuous stress field description breaks down due to the discreteness of the lattice. It is of the order of several lattice constants and about one order of magnitude above the atomic scale (∼10–50 Å). The fourth length-scale, and the one we focus on here, is that defined by the sizes of the nano- and microcracks that form dynamically in front of the propagating tip. Traditional continuum theory cannot be used on this scale due to the strong inhomogeneity. Smoothing the disorder by wishful

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homogeneization methods does not work for reasons to be detailed in this presentation. A strongly disordered region ahead of the crack is indeed observed experimentally (processing zone). We suggest here that on this 'dynamic scale' the *local* stress field ahead of the crack front relaxes very slowly, which gives rise to a supersonic-like local behaviour even though macroscopically the crack propagates slower than the shear wave speed. This leads to shielding of the tip from the far field energy equilibration and therefore to a far-from-equilibrium process. The mesoscale-dominated dynamics do not invalidate the long range continuum quasi-static calculations, as long as the latter are applied not to the bare crack tip, but rather to the tip 'dressed' by the processing zone.

Starting from the idea that the tip responds to the local stress, an equation of motion is derived from first principles. The resulting dynamic equation is solved exactly and analysed. A rich propagation is found: the crack either propagates at a steady state speed, which can be predicted from material properties, or the speed oscillates periodically. Which mode is chosen depends on one material parameter. Possible sources of noise are discussed next and it is shown that noise can strongly modify the dynamics into: quasi-periodic propagation, intermittent propagation, or a range of noise-driven steady states. The analysis of these behaviours is outlined and future directions are suggested.

1. Introduction

The effort to understand the dynamics of crack propagation is nearing the end of its eighth decade and, although experimental measurements are becoming ever more accurate, the theoretical understanding is still far from complete. The need for better materials and smarter designs has usually sufficed in other fields to rapidly clinch fundamental understanding. Yet the physics of fracture is still a puzzle and, if anything, better measurements seem to only invalidate traditional approaches rather than lead to better models. Quite a few fundamental issues have stubbornly resisted resolution. For example, many observations made on quasi-two-dimensional systems indicate that running cracks reach a limiting steady-state propagation velocity of about half the shear wave speed (SWS). Predictions of the limiting velocity date back to Mott [1], who used an energy balance argument, originally proposed by Griffith [2], to suggest the existence of such a velocity. Later [3], this velocity was proposed to be the Rayleigh wave speed (RWS), which is the speed at which surface waves propagate along the (idealized) crack surfaces behind the crack front. The reason for the factor of two discrepancy between predictions and measurements is not explained within the paradigm of energy balance considerations. Another poorly understood issue concerns the physical mechanisms for crack initiation and arrest and the reason for the stress hysteresis between the two [4,5]. Lattice trapping [6] on atomic length-scales is related to this issue and an initial understanding of this phenomenon in the microscale starts to emerge [7]. However, this aspect of the problem remains unexplained on mesoscopic and macroscopic scales. Another puzzle, which came to the fore in recent years, is the origin of the observed periodic-like oscillations of the propagation velocity in amorphous polymeric materials such as PMMA [8,9] and the relation between this observation and the properties of the material. Finally, an important problem concerns the relation between material properties and the morphology of the fracture surfaces that are left behind. These surfaces are rough on many length-scales, which characterizes them as fractal or self-affine. The current interest concerns the question whether the roughness can be related to the material toughness, if yes how, and how do the propagation dynamics affect the roughness. Partly as a result of the above problems, there is at present no consensus on the form of the equations of motion that govern continuum fracture dynamics. The growing body of experimental results seems only to add more unfitting pieces to the puzzle, a state of affairs that usually points to a problem with the basic understanding of the physical phenomenon.

Most of the approaches to model this problem were based, whether explicitly or implicitly, on quasi-static and/or equilibrium concepts to derive the dynamics of the propagating crack. Rather than resolving the above and related problems, the resulting models seem to emphasize the inadequacy of traditional approaches in tying all the loose ends. It appears that there are two main reasons for this situation. The first is experimental: Data on fast fracture processes are still restricted mostly to post-mortem measurements, while the process itself is too fast to capture by many of the current experimental devices. As a result, on-the-fly analyses are still few and far between and only recently there emerged experiments where the dynamic process can be continuously monitored. The second difficulty, however, is in this author's opinion, the main culprit for the theoretical dire straits: Fracture dynamics are governed by *different physical mechanisms on different length- and time-scales*. This aspect, although appreciated in principle, never received the full attention that it deserves and therefore it is worth some elaborating on. In ideally brittle propagation, the crack tip is atomically sharp, which means that atomic potentials are significant. This points to the relevance of the physics on scales of 5–10 Å. For large length-scales (>μm), continuum linear elasticity seems to do a good job in describing the stress field and the far-away elastic energetics. This is basically because, as mentioned above, fractures grow slower than any of the bulk speeds of sound. Since the bulk stress relaxes to its static value at a rate that is faster than the propagation rate the macroscopic system can be considered quasi-static in a frame of reference that moves with the crack front. This is also the basis for the seeming validity of utilizing energetic approaches for the far field. However, between the atomic and the continuous there are at least two more relevant length-scales. One is that of the cohesive zone. This is the region where the continuous stress field description breaks down and the stress-divergence needs to be modified due to the discrete lattice effects. This length-scale is of the order of several lattice constants, $\sim$10–50 Å.

The fourth length-scale, and the one we focus on is defined by the sizes of the microcracks that form dynamically in front of the propagating tip. In the following, the term microcracks will be used generically and the sizes involved can range from a few tens of nanometers to microns. Now, since this scale is larger than the cohesive zone, one could presume the continuous description should hold. Although correct in principle, continuum theory is difficult to apply due to the strong disorder, which translates into many small complex boundaries. These boundaries need to be taken into consideration when solving for the continuous stress and strain fields, which is an impossible task in practice. This defines the *mesoscale* regime which consists of a window of length-scales that is well above the atomic to be considered continuum, but still far below the macroscopic regime where the medium is homogeneous. An attempt to gloss over the disorder by simple homogeneization is doomed to fail for the dynamical problem. Microcracks that nucleate from vacancies or dislocations form a region whose size can be up to few microns in front of the moving tip. This region is called the processing zone (PZ) and is frequently observed experimentally.

The main contention here is that the macroscopically observed dynamics of crack propagation are governed by this disorder and therefore depends crucially on the physics in this mesoscale. Basically, the microcracks in the PZ scatter shear waves (and other sound waves) on a wide range of wavelengths and therefore suppress the local stress relaxation rate. Being strongly disordered, the speed of sound in the PZ will be then markedly lower than in the homogeneous material. The attenuation of the shear waves becomes more pronounced with increasing microcrack density and is therefore expected to be strongest very close to the tip. Since it is through the various sound waves that the stress and strain relax, it is suggested here that on this 'dynamic scale' the stress field relaxes sufficiently slowly very close to the crack tip so that the actual propagation becomes swifter than the *local* stress relaxation rate. This gives rise to a supersonic-like behaviour in spite of the fact that macroscopically the crack still moves well below the speed of sound. It should be emphasized that the unusual short-range-dominated dynamics do not invalidate the long range continuum energy calculation as long as it is remembered that these calculations apply not to the bare crack tip, but rather to the tip 'dressed' by the processing zone.

A few dynamical models for crack propagation were proposed recently for the propagation dynamics that account for the reduced limiting speed [10]. Generally, these models consist of writing down a phenomenological equation of motion that treats the tip as a massive particle with inertia and dissipation. The connection of such a phenomenological approach to the local physics that govern the dynamics on the length-scales of the PZ is far from clear. Moreover, in view of the nonlinear behaviour that ensues on various length-scales it is not very likely that the right dynamic equation can be found by this approach.

Very close to the tip, the continuous stress field drops as an inverse square root (e.g., in mode III fracture or the Yoffe solution for mode I [11], see below). The angle-independent prefactor of the inverse square root, termed the stress intensity factor K, is defined via

$$\sigma_{\alpha\beta} = K f_{\alpha\beta}(\theta)/\sqrt{2\pi r}, \tag{1}$$

where $\sigma_{\alpha\beta}$ is the stress tensor element, r is the distance of a point in the plane from the moving tip and $f_{\alpha\beta}$ is a tensor element that depends only on the azimuthal angle θ, and which can be obtained using formal expansions in a complete set of basis functions (e.g., Legendre polynomials). Traditional treatments regard the stress intensity factor as a key quantity that is presumed to be modified as the crack moves. All dependence on time, propagation speed, the cracking mode (I, II or III), and the propagation history are usually lumped into K and much effort goes into understanding the nature of this dependence. We argue here that this approach may miss quite important underlying physics. The form of Eq. (1) is based strongly on the assumption that the propagation is quasi-static. Namely, it is usually taken for granted that the stress field near the tip relaxes to its static value sufficiently fast such that the stress field has essentially the static form when viewed in the tip's moving frame. Under this assumption the crack tip coincides with the singularity of the stress field ($r \to 0$ in the continuous description). While the aforementioned observations of macroscopic subsonic propagation seem to support this view, there is a leap of faith from the fact that the crack propagates subsonically on the continuum scale to the conclusion that the crack dynamics are dictated by the far-field energetics. A necessary condition for balancing the elastic and surface energies is that the flow of information to the tip about the changing stress field is *locally* faster than the propagation rate. However, existing evidence [12] strongly suggests that this is not the case. In particular, the measurements in [12] showed a significant reduction in the local stress relaxation rate after crack arrest, with the relaxation time being of order of tens of μs. To appreciate the significance of this observation, recall that relaxation of the stress field to its static value in the homogeneous bulk would have been almost an order of magnitude faster! Due to this slow local relaxation, the tip is in fact shielded from the far field by the highly inhomogeneous PZ. This shielding from the global behaviour, elevates the importance of the near-tip mesoscale physical mechanisms that dominate what becomes a strongly nonequilibrium process. Ultimately, it is the nonequilibrium nature of the problem that invalidates the energetic considerations: With cracking a new boundary forms in the continuous material and the information that such a boundary has formed travels ahead of the crack at a much reduced speed. Comparing the slow local relaxation rate with the observed propagation speed, it is seen that locally the crack propagates at a speed that is higher than the relaxation rate. Thus, the tip travels ahead of the wave front that relaxes the field and the stress near the tip is *lower* than the static value. It is emphasized that the supersonic-like scenario applies in spite of the observed macroscopic subsonic propagation. Other than calling into question the assumptions that underlie traditional treatments, a locally supersonic propagation casts doubt on the utility of the stress intensity factor in Eq. (1). Nevertheless, since K has been measured in many experiments, we will provide a mapping between the results obtained here and this quantity to allow for a re-interpretation of existing experimental results in the context of the proposed theory.

In this presentation, we derive from first principles, rather than assume, the form of the equation of motion of the crack tip. Starting from the idea that the tip responds to the local stress field, and that the stress field relaxes slowly relative to the homogeneous bulk, we first obtain a time-dependent equation for the evolution of the tip stress. This equation is nonlinear but it is amenable to detailed analysis and even to a formal solution. Further postulating a generic (not specific) form of the local velocity-stress response, the analysis can be made explicit and quantitative results are obtained. The velocity-stress response can be inferred both from experimental measurements and from a recent first-principles calculation on atomic systems [7]. This parameter-free approach turns out to give rise to rich propagation dynamics even in the absence of noise. Noise, however, is inherent to the fracturing process even at zero temperature, and should not be left out in any serious attempt to model the fracture process. The noise can originate from various origins: First, on atomic scales, the atomic bond breaking events are very violent and excite vibrations which in turn may (or may not, see below) spread in the system. Second, the distribution of the microcracks in the PZ introduce fluctuations because of the local change in material properties that the tip experiences. We discuss effects of noise and show that it can strongly modify the dynamics. Specifically, depending only on one material parameter, λ, and the noise characteristics, the crack tip can display various behaviours: Steady propagation at a constant limiting velocity, periodically oscillating propagation, intermittent propagation, or a range of noise-driven steady states and quasi-periodicities.

2. Kinetics of crack propagation and equation of motion

The crack propagates through continuously generating a new surface. The stress field adjusts to the new boundary that forms via sound waves that travel in the medium. Within the traditional picture the stress field relaxes at the speed of the Rayleigh waves which travel along the newly formed surfaces. This picture is attractive due to its simplicity but it suffers from a shortcoming in that it is not consistent with observations that the surfaces that are left behind are rough on a large range of length-scales. With such roughness it is difficult to understand how exactly the surface waves can propagate freely. This is because roughness acts as disorder and would tend to enhance nonlinear effects that can range from localization to nonlinear vibrational excitations on these scales. Moreover, the interaction of the surface waves with the vibrating medium is hopelessly complex to analyse and a simplistic energy-based argument can hardly do any justice to it, let alone predict the rate at which energy is transferred along the surface.

A careful consideration of the atomic motion near the tip [7] reveals that the relevant sound waves that propagate the relaxation from the tip to the rest of the system are in fact the local shear waves. In the following we assume, for simplicity, that the local SWS, c, is constant and independent of the degree of inhomogeneity. This assumption is made to simplify the derivation of the equation of motion, but is not essential. This speed is well below the bulk SWS, as discussed in the introduction. It will be shown in a different report [13] that, when this assumption is lifted, one can still, write, solve, and analyse the equation of motion with the analysis being only little modified.

To derive the dynamic equation for mode I, consider the expression for the stress field just ahead of the moving crack, as derived in [11]

$$\sigma = \sigma_\infty \frac{\zeta + a}{\sqrt{\zeta(\zeta + 2a)}}, \tag{2}$$

where ζ is the distance from the tip, σ_∞ is the mode I tensile stress that is applied perpendicular to the propagation axis far away from the crack, and a is the crack's length, which is assumed to be constant. We rescale in the following the stress and write it in units of $\sigma_\infty, \sigma \to \sigma/\sigma_\infty (> 1)$. The above solution assumes implicitly that the singularity of the stress field occurs at the tip, $\zeta = 0$, and increases as $\sigma_{\mathrm{tip}} \sim \zeta^{-1/2}$ as the crack tip is approached, i.e., when $\zeta \ll a$. The singularity is cut off, of course, on the scale of the cohesive zone (~10–50 Å), which is assumed to be well below the (meso)scale of interest here. While at subsonic propagation (tip velocities below the local SWS) the stress field singularity coincides with the crack tip, the local supersonic propagation separates the two. This is because the stress field at the tip cannot relax sufficiently fast and therefore the singularity of the field lags behind. This separation can persist even when the tip velocity drops below the SWS since it takes some time for the shear wave to catch up with the tip. We can now re-interpret Yoffe's solution if ζ is regarded as the separation distance between the crack tip and the singularity in the stress amplitude. Eq. (2) gives then the stress at the crack tip when the tip is exactly a distance ζ ahead of the singularity. Thus, the stress at the tip, $\sigma_{\rm tip}$, is obtained from Eq. (2) by putting $\zeta = (l - ct)\Theta(l - ct)$, where l is the tip's position along its path and $\Theta(l - ct)$ is the Heavyside function which ensures that when the shear wave does overtake the tip $(l - ct \to 0^+)$ then the stress converges to the static value in the moving frame and remains at that value for all $l < c$.

The behaviour of the crack propagation for $l < ct$ $(\Theta = 0)$ is well studied [14] and need not be repeated here. It suffices to focus only on the case $\Theta = 1$ (i.e., when the tip precedes the singularity). For propagation dynamics that mix both $\Theta = 0$ and $\Theta = 1$ one simply pieces the analytic solutions together along the path. Note that the analysis presented here does not require that the tip propagate at a straight line. Rather, the local tip velocity, $v = {\rm d}l/{\rm d}t$, should be measured along the crack path irrespective of the path's geometry. Taking the time derivative of Eq. (2), we obtain

$$\frac{{\rm d}\sigma}{{\rm d}t} = -\frac{{\rm d}\zeta}{{\rm d}t}\frac{a^2}{[\zeta(\zeta + 2a)]^{3/2}}. \tag{3}$$

In what follows, the subscript from the tip stress can be omitted and use σ instead, except where it may lead to confusion with the stress field. Using Eq. (2), express ζ in terms of σ and rewrite ${\rm d}\zeta/{\rm d}t = v - c$. Recalling then that $v = v(\sigma)$, Eq. (3) can be manipulated to yield

$$\frac{{\rm d}\sigma}{{\rm d}t} = -\frac{1}{a}[v(\sigma) - c](\sigma^2 - 1)^{3/2}. \tag{4}$$

This is the equation that defines the time evolution of the stress tip as the crack propagates. Now solve Eq. (4) by quadratures and, following a straightforward manipulation, the time dependence of σ can be obtained:

$$t = -\frac{a}{c}\int^{\sigma(t)} \frac{{\rm d}s}{u(s)(s^2 - 1)^{3/2}}. \tag{5}$$

In this relation, $u(\sigma) = [(v(\sigma) - c)/c]$ is the (dimensionless) reduced velocity. Relation (5) is in fact the formal solution to the kinetics of the tip stress and forms the basis of the analysis. From this solution $\sigma(t)$ is found which can be substituted into $v(t)$ to find $v(t) = v(\sigma(t))$. The latter can, in turn, be integrated to give $l(t)$ and $\zeta(t)$. It is important to emphasize that the solution (5) is an exact derivation from the Yoffe solution and the only assumption that goes into it is that the local propagation rate depends on σ alone. When v depends on other variables (e.g., explicitly on time) the equation of motion (4) still holds true but Eq. (5) should be modified.

The stress intensity factor can now be re-interpreted for its dependence on time, velocity, and history, in terms of the above solution: Experimental setups usually measure the stress at some small fixed distance d_0 ahead of the propagating crack tip (for clarity, ignore the azimuthal dependence). This distance usually depends on the measurement method. Since it is measured from the moving singularity, this distance is in fact $\zeta = d_0 + c\int u(t)\,{\rm d}t$. By writing the dimensionless stress at that point both in terms of the stress intensity factor and the Yoffe expression, there results

$$\sigma(d_0) = \frac{\zeta + a}{\sqrt{\zeta(\zeta + 2a)}} \equiv \frac{K}{\sigma_\infty\sqrt{2\pi d_0}}. \tag{6}$$

A simple manipulation of the two rightmost parts of this expression leads then to the time and velocity dependence of the stress intensity factor,

$$K = K_0\frac{1 + a/\zeta}{\sqrt{1 + 2a/\zeta)}}, \tag{7}$$

where $K_0 = \sigma_\infty(2\pi d_0)^{-1/2}$ is a constant. Thus, the stress intensity factor depends explicitly on $\zeta(t) = [l(t) - ct]\Theta(l - ct)$, and we can now understand the origin of the observed time and history dependencies. We suggest that relation (7) can be helpful in checking this theory against existing experimental measurements of K.

3. The equation of motion: Analysis and fracture kinetics

Now proceed to analyse in detail the solution to the equation of motion, (5). To make progress, we need some information for the form of the local velocity response to the stress, $v(\sigma)$. Many reports in the literature suggest that the speed vs. stress behaviour is hysteretic with two material-dependent thresholds: σ_h, above which crack propagation initiates and $\sigma_l < \sigma_h$, to which the stress has to drop for the crack to halt (arrest) [4,5]. It has also been observed that for stresses higher than σ_h, the velocity increases very slowly as the stress is increased [15]. This also agrees with reports that, in this regime, considerable changes in K seem to have little effect on the velocity [16]. All these observations suggest a generic form of $v(\sigma)$ which is shown in Fig. 1. Along the upper branch of the hysteretic curve the velocity increases slowly with σ, while along the lower branch the velocity can be either identically zero (complete arrest) or very small (creep). As will become clear below, the dynamics of the propagation is essentially determined by the location of the SWS, c, relative to the speed v_l on the response curve. For later reference, it is convenient to classify the behaviour in terms of the dimensionless material dependent parameter $\lambda \equiv c/v_l$.

Now turn to a discussion of noise-free propagation. This term does not mean that fluctuations in the local parameters and fields are completely absent, but rather that these are negligible on time- and length-scales that are relevant to: (i) the equation of motion, and (ii) to measurement times. Differently put, there may well exist rapid fluctuations on short time-scales, but we assume in this section that these either cannot be observed or that they do not affect the slower mesoscale dynamics (fast versus slow degrees of freedom). We consider then the propagation process within the context of an effective continuum (EC) whose properties are different from the bulk material. The EC is a good description when the number of microcracks is large enough to effect rapid fluctuations and the microcrack sizes are sufficiently small so that an individual fluctuation due to a large microcrack cannot be detected. When these conditions are met, Eqs. (3)–(5) provide a good model for the motion of the tip. The alternative scenario, when large microcracks exist and can give rise to measurable stress fluctuations at the tip, will be discussed in detail later on when noise is incorporated into the model.

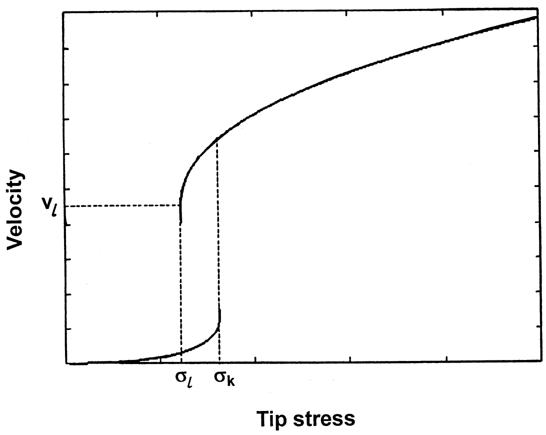

Fig. 1. A generic form of the constitutive relation $v(\sigma)$.

In principle, propagation in the noise-free context can be divided into two cases that lead to different dynamics:

3.1. $\lambda > 1$

In materials where this relation holds, the value of c corresponds to a point, $F = (\sigma_c, c)$ that is located on the upper branch of $v(\sigma)$. If we inspect the integrand on the right-hand-side of Eq. (5), we can see that it is regular at the point F. The behaviour at the vicinity of this point can be found then by a standard linear analysis, which readily yields

$$|\sigma - \sigma_c| \cong \text{Const. } e^{-\gamma\tau}, \qquad (8)$$

where

$$\gamma \equiv \frac{(\sigma_c^2 - 1)^{3/2}}{a} \left(\frac{dv}{d\sigma}\right)_{\sigma_c}.$$

In the above expression, we observe that the derivative of $v(\sigma)$ is continuous and positive in the neighborhood of F, which means that γ is positive. Thus, if the system is at a state near the point F, it will converge to it at a rate that is determined by γ.

It follows that F is a *stable fixed point* of the equation of motion for all $\lambda > 1$. In the units that have been chosen above we have $\sigma_c > 1$ and $0 < (dv/d\sigma)_{\sigma_c} < 1$ [15,16]. To estimate the value of γ we recall that: (i) the continuous description pertains to length-scales larger than the cohesive zone, $r_0 \sim 5$ nm, and (ii) the typical velocity of the steady state in, say, PMMA is $c \sim 500$ m/s. Thus, typical times are of order $r_0/c \sim 10^{-11}$ s. It follows that the relaxation to the steady state from a state in the vicinity of F is so swift that small fluctuations are not only practically invisible by current measuring techniques, but they also occur at time-scales that are only two or three orders of magnitude above atomic vibrations. A typical relaxation of the system to the steady state is shown in Fig. 2.

The solution represented by the stable point F should be interpreted as a settling of the crack tip into a steady propagation rate, c, during which the local stress at the tip is constant at σ_c. We propose that this is the steady state which is frequently observed in experiments. Special attention should be given to the fact that the steady-state speed is exactly the local SWS c. This has an important implication in interpreting experimental data. It also differs from the traditional view that this speed is the same as the speed of the Rayleigh waves along the fracture surfaces.

Another piece of information from the literature pertains to the location of F on the curve of $v(\sigma)$. Immediately after crack initiation, the stress intensity factor drops and then stabilizes at a fixed value [4,5,15]. This phenomenon has been attributed to inertial effects [4] but, in view of the above analysis, this may suggest that in those materials the system slides down along the upper branch after crack initiation, until it reaches the fixed point F and the stress saturates to σ_c. It thus means that in these systems $\sigma_h > \sigma_c > \sigma_l$. Moreover, the fact that this decrease is observable on time-scales that measurements can detect indicates that either or both scenarios occur: (i) the interval $\sigma_h - \sigma_c$ is quite large, and (ii) $dv/d\sigma$ is indeed small along the upper branch. A detailed analysis is suggested for the time dependence of the drop in the tip stress after initiation is likely to yield the actual form of $v(\sigma)$ along the upper branch *near and above* the stable fixed point. Note, though, that in materials where $\sigma_c > \sigma_h$ the system will slide *upward* after crack initiation, which again can be utilized to chart the constitutive v–σ relation below σ_c.

A word of caution: the above steady-state solution and its interpretation are based on noise-free analysis. It is shown below, however, that noise can drive the systems into a different steady state, whose measured limiting velocity is lower than c. Therefore, the limiting velocity data should be carefully interpreted, taking into consideration the history of $v(t)$.

3.2. $\lambda < 1$

The second important case is $c < v_l$ and it is useful to consider it qualitatively first. Suppose that at a given moment the crack is propagating at some speed, v, that corresponds to a particular state (point) along the upper branch. Since on the upper branch $v > c$, the tip propagates faster than the relaxation of the stress field and the tip stress steadily decreases. This can be best seen from relations (4) and (5) that show that the system will slide down the upper branch with both the tip velocity and stress dropping simultaneously. Unlike the previous case, since there is no fixed point on the upper branch, the system is unable to converge to a steady state and, on reaching the left end of the upper branch, $F_l = (\sigma_l, v_l)$, it goes over the edge and drops to the lower branch. On the lower branch, the crack settles either into a temporary arrest (if $v_{lb} = 0$) or a slow creep (if $v_{lb} > 0$). Now, the shear waves that have been lagging behind start to catch up and the stress at the tip builds up. The increase in stress pushes the system up along the lower branch until it reaches the crack-initiation value, σ_h. At this moment the crack jumps back to the upper branch and the crack bursts away again. Propagating now at $v > c$, the crack tip is faster than the stress relaxation front and the tip moves away from the stress singularity, causing both the tip stress and the tip velocity to decrease again. The system then slides down the upper branch and the cycle repeats itself. This repetition gives rise to a periodic propagation behaviour.

With this picture in mind, calculate the characteristic times involved in the periodic process. Denoting by T the full cycle period, this period consists of the time that the system spends moving from σ_h to σ_l along the upper branch, and then in the opposite direction along the lower branch. Using relation (5) T can be written as

$$T = \frac{a}{c}\int_{\sigma_l}^{\sigma_h}\left[\frac{u_{\rm ub}^{-1}}{(s^2-1)^{3/2}} - \frac{u_{\rm lb}^{-1}}{(s^2-1)^{3/2}}\right]\mathrm{d}s \qquad (9)$$

where $u_{\rm ub}$ and $u_{\rm lb}$ are, respectively, the reduced velocities along the upper and lower branches. Note that $u_{\rm lb}$ is negative ($v < c$) and therefore both the terms in the integrand are positive. A typical periodic state is shown in Fig. 2. A complete crack arrest on the lower branch ($v_{\rm lb} = 0$) corresponds to $u_{\rm lb} = -1$, while creep corresponds to $u_{\rm lb} = -1 + \varepsilon$, where $\varepsilon > 0$. For illustration, consider the following constitutive relation

$$v_{\rm lb} = 0; \qquad v_{\rm ub} = \frac{v_0(\sigma^2 - A)}{\sigma^2 - 1} \qquad (10)$$

where $A = (v_0 + c/v_0)$. The first term on the right hand side of Eq. (9) yields

$$T_{\rm ub} = \frac{a}{2c}\ln\left(\frac{\alpha - y_l}{\alpha - y_h}\frac{\alpha + y_h}{\alpha + y_l}\right), \qquad (11)$$

where $\alpha = (v_0/c)^{1/2}$, $y_k = -\sigma_k/(\sigma_k^2 - 1)$ and k is either h or l. The second term on the right-hand side of Eq. (9) yields

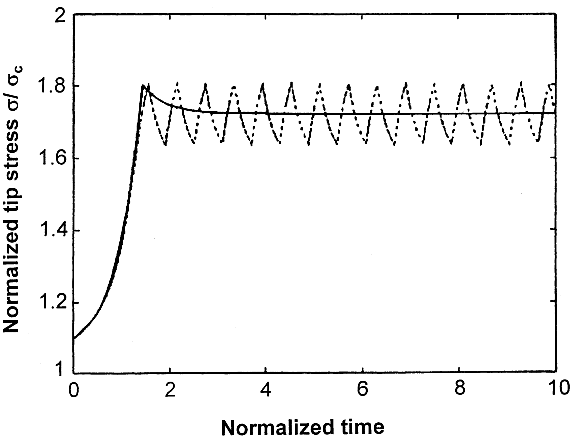

Fig. 2. The stress history in the steady and periodically oscillatory states.

$$T_{0,l} = \frac{a}{c}\int_{\sigma_l}^{\sigma_h}\frac{\mathrm{d}s}{(s^2-1)^{3/2}} = \frac{a}{c}\left(\sqrt{\sigma_l y_l} - \sqrt{\sigma_h y_h}\right). \qquad (12)$$

Thus the entire cycle period is

$$T = T_{\rm lb} + T_{\rm ub} = \frac{a}{c}\left(y_h - y_l + \ln\frac{\sqrt{\alpha - y_l}}{\alpha - y_h}\frac{\alpha + y_h}{\alpha + y_l}\right). \qquad (13)$$

For $\varepsilon \ll 1$ (slow creep) expand the expression for the time spent on the lower branch and find that to first order in ε

$$T_l = T_{0,l}(1 + \varepsilon).$$

When ε is of order 1 the lower branch does no longer describe slow creep, but rather, the lower branch increases appreciably. The system may then experience not only two velocities for a given value of the stress but also two stresses for a given velocity, which happens when $v(\sigma_h) > v(\sigma_l)$. The occurrence of this interesting situation may in fact be supported by observations in PMMA where low values of velocity fluctuations seemed to be well above zero [9]. This scenario can be studied straightforwardly along the lines outlined in the above analysis. We avoid going down this path in order to keep this presentation clear.

This analysis is not complete without mentioning the marginal case of $\lambda = 1$, which is relevant only theoretically in the context of absolute noise-free propagation. In this case the point F is at $F_l = (\sigma_c = \sigma_l, c = v_l)$ and the propagation dynamics depends strongly on the behaviour of $(\mathrm{d}v/\mathrm{d}\sigma)_F$, when F is approached from above. If the slope at F_l is finite at this point, the previous analysis applies unchanged and the approach to the fixed point follows relation (8). An approach to F_l from below is impossible because $\sigma < \sigma_l$ corresponds to a state on the lower branch. Thus this is a 'one-sided stability'. There is no apriori reason, however, why $(\mathrm{d}v/\mathrm{d}\sigma)_F$ should be finite. If the slope diverges the dynamics of the crack tip are sensitive to the specific rate of the divergence. For example, consider the following form for $v(\sigma)$ near σ_l.

$$v = v_l + v_0\left(\frac{\sigma}{\sigma_l} - 1\right)^n;$$

$$u \sim \frac{v_l - c}{v_0} + (\sigma - \sigma_l)^{\eta}, \quad (14)$$

where $0 < \eta < 1$ is a dimensionless number and v_0 is a prefactor with dimensions of velocity. This relation can be regarded as an approximation to a frequently occurring functional form in fracture

$$v = v_l + v_0\left(1 - e^{-(\sigma/\sigma_l - 1)^{\eta}}\right); \quad \sigma \geqslant \sigma_l. \quad (15)$$

The behaviour near the fixed point can be calculated from Eq. (5) and yields

$$\sigma - \sigma_c \sim (\tau_o - t)^{1/(1-\eta)} \quad (16)$$

and

$$u \sim (\tau_o - t)^{\eta/(1-\eta)}, \quad (17)$$

where $\tau_0 > t$ is a characteristic time whose value is determined by the properties of the material under consideration. This solution describes again a convergence of the velocity of propagation to $c = v_l$ but, unlike the former case, the convergence now is at a *power law* rate rather than exponentially. In this case, F_l can be described as *marginally stable*. This marginal state is extremely sensitive to noise because when the crack propagation is close to the fixed point (σ_l, v_l) very small fluctuations can knock the system over the edge down to the lower branch. Therefore, this scenario, although theoretically interesting, is not very likely to be observed in real situations.

To summarize this section, the dynamics of the tip in the noise-free regime is characterized either by a steady-state propagation at the limiting velocity c, or by a periodically oscillating propagation rate with a period T, which has been found explicitly. It is emphasized that there is only one material dependent parameter, λ, that controls which of these modes will come into play. The marginal propagation mode, although existing in principle according to the equation of motion, should be practically impossible to detect due to the inevitable noise that always exists.

4. Kinetics of Mode III propagation

The above treatment is extended to Mode III propagation [17]. As it turns out, there is no quantitative difference in adapting the calculations to this mode. Only the explicit form of the solution for $\sigma(t)$ changes between the two modes. Again, start from the spatial dependence of the stress field, which, for mode III propagation is

$$\sigma_{yz}|_{y=0} = K^* / \sqrt{2\pi\left(x - \int c \, dt\right)}. \quad (18)$$

Here the tip motion is assumed, for brevity, to occur along the x-axis as defined by the symmetry of the mode-III boundary conditions, and $\int c \, dt$ is the position of the stress singularity. Since the propagation is supersonic the tip is ahead of the stress singularity and the above expression can again be reinterpreted as describing the tip stress in terms of its distance from the field singularity,

$$\sigma_{tip} \equiv \sigma = \frac{K^*}{\sqrt{2\pi\left[l(t) - \int c \, dt\right]}}. \quad (19)$$

The term under the radical is the location of the tip in the frame of the moving stress field singularity, which propagates at speed c. The value of c may, or may not, be constant, an issue that is discussed in a different report [13]. Differentiating Eq. (18) with respect to time we now have

$$\dot{\sigma} = -\left(\frac{K^*}{2\sqrt{2\pi}}\right) \frac{v - c}{2\left[l(t) - \int c \, dt\right]^{3/2}} = -\frac{\pi\sigma^3 (v - c)}{K^{*2}} \quad (20)$$

Inverting this relation, we obtain the exact solution for $t(\sigma)$:

$$t - t_o = -\left(\frac{K^{*2}}{\pi}\right) \int_{\sigma(t=t_0)}^{\sigma(t)} \frac{ds}{s^3 [v(s) - c]}. \quad (21)$$

Presuming that the constitutive local velocity-stress relation remains the same as for mode I, the above analysis carries over to this case with essentially similar conclusions:

(1) For $\lambda > 1$, Eq. (21) has a fixed point on the upper branch at $v(\sigma_c) = c$. A linear stability analysis around the fixed point yields that it is stable with

$$\delta\sigma \sim e^{-\gamma t} ; \qquad \gamma = \frac{\pi\sigma_c^3 (\partial v/\partial\sigma)_{\sigma_c}}{K^{*2}}$$

Therefore, when the system is on the upper branch, it flows to the fixed point both from $v > c$ and $v < c$. As in mode I, the stable fixed point corresponds to the crack propagating at a steady-state velocity c.

(2). For $\lambda < 1$, the propagation consists of a series of relaxation cycles. The period of oscillation can be calculated explicitly again and turns out to be:

$$T = \frac{K^{*2}}{c\pi} \int_{\sigma_l}^{\sigma_h} \left(\frac{1}{u_{\mathrm{ub}}} - \frac{1}{u_{\mathrm{lb}}} \right) \frac{\mathrm{d}s}{s^3}, \quad (22)$$

where u_{ub} and u_{lb} are the upper and lower branch reduced velocities, as before.

5. Dynamics in the presence of noise

Now proceed to address noise and its effects on the above results. First, identify the relevant source of noise in this system. As mentioned in the introduction, even in the absence of external (e.g., thermal) fluctuations, the medium vibrates strongly on atomic wavelengths. The reason is that the crack grows through a series of bond breaking events that give rise to violent vibrational excitations. These excitations are nonlinear due to two effects: (1) The nonlinearity of the interatomic potentials (recall that just ahead of the tip the atoms are strained to well beyond the linear Hookian approximation). (2) Due to the disordered structure on the atomic scale in the cohesive zone. The nonlinearity manifests in localization and coupling between different modes. Moreover, whereas linear excitations (phonons) propagate smoothly from the excited tip, the lattice nonlinearity leads to nontrivial 'leaking' of energy from that zone. This, in turn, gives rise to an energy build-up in front of the propagating crack tip, as has indeed been observed in simulations [18]. However, having said all that, these effects are expected to dominate only on length-scales that are at most a few nm and therefore cannot persist to higher length-scales mostly due to localization effects. It follows that these vibrations are irrelevant on the length-scales discussed here and noise in the mesoscale comes mainly from the spatial and statistical size distribution of microcracks. Remember that by microcracks we refer to inhomogeneities that occur in the PZ and which are much smaller than the main crack. The material inhomogeneities form and grow in response to the enhanced strain in front of the propagating crack and therefore their size and spatial distributions not only play a significant role in the dynamic propagation but are also determined self-consistently by the very same dynamics. Here we take for simplicity these distributions as given and proceed to consider the implications within the current theory. To keep the model simple, it is assumed in the following that $v(\sigma)$ remains unchanged on the time and length scales that are relevant to measurements, and only the local stress at the tip is affected.

What happens physically is that the main crack propagates until it encounters a microcrack, whereupon the tip stress and position change discontinuously as follows: Suppose the main crack joined the microcrack at a point B along the boundary of the latter. After the association event propagation will resume from a point along the boundary of the microcrack which is *different* than B. So the location of the tip jumps discontinuously from one point along the microcrack boundary to another. At the same time, the tip stress drops discontinuously because the stress at the new point of propagation is inevitably lower than the stress at the original point.

Distinction will be made between two regimes: (1) When the microcrack sizes are sufficiently small so that the positional changes are undetectable on measurable scales but stress fluctuations can be observed. In this regime only stress drop events need be considered and the small jumps in the tip's position can be ignored. (2) When there are sufficiently big microcracks so that even the positional jumps can be detected experimentally. In the latter regime, one needs to resort to a detailed statistical analysis, which is a straightforward extension of the analysis to be presented below. No attempt will be made in this presentation but rather defer it to a later report. As it turns out, even the first regime reveals a rich behaviour, as is shown in the following.

6. Effects of fluctuations in the steady state

When $\lambda < 1$, fluctuations only smear the periodicity with the extent of the smear depending on the size and frequency distributions of the fluctuations. An explicit quantitative analysis of this regime is left for another report. Here focus is made only on a few cases when $\lambda > 1$, which yield a somewhat richer behaviour. The dynamics for $\lambda > 1$ are determined by several issues: (1). The noise amplitude, A; (2) The value of $\Delta_0 = \sigma_c - \sigma_l$, namely, the difference in stresses between the fixed and arrest points; (3) The frequency of occurrence of the fluctuations, ω (which may well be distributed over a wide range of frequencies). A central quantity in all that follows is the probability density of fluctuations in the tip stress amplitude, $P_A(\delta\sigma = \sigma_c - \sigma)$ and it is therefore convenient to classify the behaviour by the amplitude and frequency of occurrence of the fluctuations. This classification spans an interesting state diagram in the A–ω plane.

6.1. $\omega \ll \gamma$ and $A \ll \Delta_0$

Consider first a very low occurrence frequency of fluctuations in σ, $\omega \ll \gamma$, where γ is defined in Eq. (8), and assume that there are no fluctuations whose amplitude is greater than Δ_0, namely,

$$\mathrm{Prob}(A > \Delta_0) = \int_{\Delta_0}^{\infty} P_A(x)\,\mathrm{d}x = 0$$

The more dilute the spatial distribution of microcracks, the longer the delay times between successive fluctuations and the smaller is the frequency ω. The smaller the sizes of the microcracks, the smaller are the fluctuation amplitudes and therefore the smaller are the values of A. Consider now what happens when a fluctuation has just kicked the system from the fixed point to a new state of stress and velocity. The new state is located on the upper branch somewhere between σ_c and σ_l. From the new point the system moves back to the fixed point by accelerating according to Eq. (21), whereupon it settles back into the steady state and awaits a new fluctuation. The probability density of the time intervals that it takes the system to settle back into the fixed point, τ, that can be calculated directly from P_A and Eq. (21):

$$P(\tau) = P_A(\delta\sigma)|\delta\dot{\sigma}(\tau)| = \frac{c}{a} P_A(\delta\sigma) \left| u_{\mathrm{ub}}(\delta\sigma) \left[(\sigma_c - \delta\sigma)^2 - 1 \right]^{3/2} \right|, \quad (23)$$

where $\delta\sigma$ is a function of τ which is obtained from Eq. (21). Thus, this case displays small fluctuations around the steady-state propagation rate, whose probability density is given by Eq. (23). At this point we can close a circle and provide a self-consistent criterion for what one can consider low occurrence frequency of stress fluctuations: For the above calculation to remain consistent, the distribution of the time intervals between two successive fluctuations θ_n, need to satisfy the relation $P(\tau > \theta_n) \ll 1$.

6.2. $\omega > \gamma$ and $A \ll \Delta_0$

We now consider occurrence frequencies of tip stress fluctuations moderately higher than γ, whose amplitudes are still below Δ_0. Physically, this means that the microcracks are still small but that they are more densely distributed in the PZ. Now, once the system has been kicked away from the fixed point, the tip is typically not allowed sufficient time to return to this state before a new fluctuation appears that further reduces the stress. To illustrate the statistical calculation, consider the following example of kinetics: The system has just jumped to the upper branch at σ_h and starts sliding down according to the equation of motion (5). After a period of time, θ_1, of smooth drop, a fluctuation of size A_1 occurs when the tip stress is at $\sigma(\theta_1)$. This causes the stress to drop discontinuously to $\sigma(\theta_1) - A_1$. From this new point, the system slides again smoothly according to the equation of motion until another fluctuation occurs. Consider the statistics of such a series of events. Of particular interest is the time, T_{u}, that the system spends on the upper branch before it reaches σ_l. To compute this time, regard the process as a time series of fluctuations of sizes A_n, each following a quiescent period of θ_n. The total time spent on the upper branch is

$$T_{\rm ub} = \frac{a}{c}\sum_{n=1}^{N} \int_{\sigma_n}^{\sigma_{n-1}-A_{n-1}} \frac{{\rm d}x}{u_{\rm ub}(x^2-1)^{3/2}}, \tag{24}$$

where $\sigma_{n=0} \equiv \sigma_h$, $A_{n=0} \equiv 0$, σ_n is the stress that the system reached starting from $\sigma_{n-1} - A_{n-1}$ following the equation of motion for a period of time θ_n, and N is the total number of fluctuations needed, starting from σ_h, to drop all the way down to σ_l. In Eq. (24), there is a difference in the kinetics above and below σ_c: For $\sigma > \sigma_c$ the system drops between successive fluctuations, thus adding to the downslide of the system. For $\sigma < \sigma_c$ the kinetics act to *increase* the local tip stress, opposing the downslide. This difference is important in that it gives rise to a continuous spectrum of noise-driven steady states which can occur only for $\sigma < \sigma_c$, as shown below. Once the system has dropped to the lower branch, the tip stress increases without interruption from σ_l to σ_h, which takes

$$T_{\rm lb} = -\frac{a}{c}\int_{\sigma_l}^{\sigma_h} \frac{{\rm d}x}{u_{\rm lb}(x^2-1)^{3/2}}. \tag{25}$$

Thus, the period of an entire cycle is $T = T_{\rm lb} + T_{\rm ub}$, whose distribution can be computed from the above expressions in a straightforward manner. The observed behaviour depends largely on the statistics of the ratio $T_{\rm ub}/T_{\rm lb}$. If this ratio is sharply distributed then the behaviour is quasi-periodic, while if it is widely distributed then the dynamics is either intermittent or chaotic. Although a detailed statistical analysis will not be carried out here, it can be seen from relation (24) that the behaviour will be sensitive to both the distributions of θ_n and A_n. An analysis, assuming the standard Weibull statistics for the distribution of the microcracks is currently under way where we study the implications on the propagation dynamics.

For a very high density of microcracks in the PZ, many fluctuations in quick succession can push the system all the way down to σ_l and over the edge to the lower branch. The crack will be observed then to propagate haltingly until it drops to the lower branch and stops. There the crack waits for the stress to build up again until a new initiation occurs. Thus, very high occurrence frequencies of small fluctuations can lead to chaotic, intermittent or even seemingly periodic behaviour, depending directly on the statistics of the noise.

6.3. $\omega \approx \gamma$ and $A \approx p\Delta_0$; $0 < p < 1$

So far, it is found that small fluctuations at low occurrence frequencies (namely, large values of θ) affect the steady state very little, while high frequencies can lead to various complicated dynamic behaviours that depend on the distributions of θ_n and A_n. Similarly, intermediate and small values of θ can be analysed. These results are not presented here. Rather, one particular regime will be mentioned which gives rise to a new phenomenon: A continuum of solutions corresponding to noise driven steady-states.

Consider a system whose material parameter is $\lambda = \lambda_0 > 1$. Suppose that at time t_0 the system is at some point σ_0 on the upper branch such that $\sigma_l < \sigma_0 < \sigma_c$. Now let a fluctuation of size A_0 jolt the system down to $\sigma_0 - A_0 > \sigma_l$. According to the equation of motion, the stress will increase until a new fluctuation arrives or until the stable fixed point is reached. If the occurrence frequency of fluctuations is not too low a new fluctuation will come along after a time θ_1 and before the system converges to the fixed point. This fluctuation reduces the stress of the tip to a new stress, $\sigma_l - A_1$, and the system starts to slide up the upper branch again towards the fixed point. On the way up, however, yet another fluctuation occurs, the tip stress drops again and so on. If the values of θ_n are typically similar to the time that it takes the tip stress to build up to the same value that it started from ($\omega \approx \gamma$). The tip stress can only build up to approximately the initial stress before another fluctuation comes along and knocks it down again. Thus the stress will be knocked about back and forth *below* σ_c around some average velocity $\lambda_{\rm measured} < \lambda_0$. Measurement of the limiting speed will give in this case a value that lies below the real SWS in the PZ.

As mentioned, the analysis for $\lambda < 1$ can be carried out along very similar lines. It reveals that

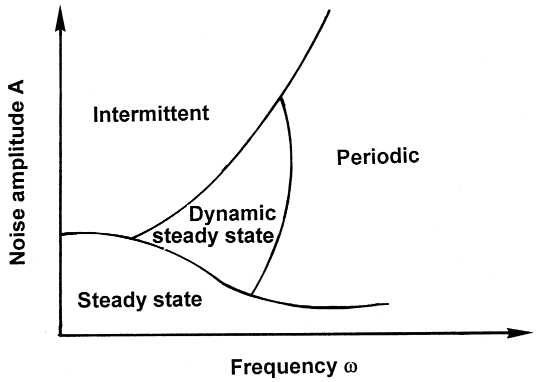

Fig. 3. The state diagram for $\lambda > 1$.

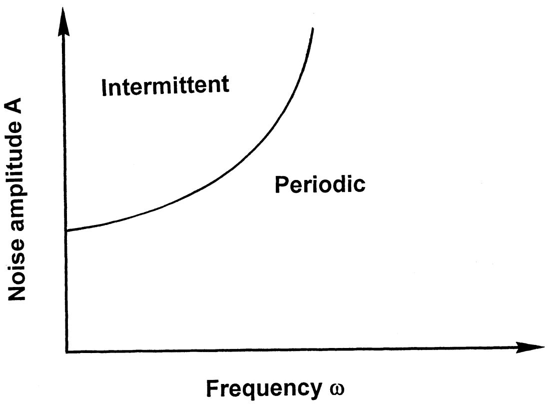

Fig. 4. The state diagram for $\lambda < 1$.

in this regime the behaviour either fluctuates about the periodic propagation mode, or it becomes intermittent. The full state siagram is shown in Figs. 3 and 4.

Note that the separation in the A–ω parameter space cannot be sharp; Large fluctuations with intermediate occurrence frequencies can lead to behaviours that range from completely chaotic, through intermittent to periodic, all depending on the interplay between the distributions of A and θ.

It should be stressed that small fluctuations always occur and therefore no proper steady-state propagation can exist for Δ_0 below the background noise level. Thus, for narrow distributions of A and ω (i.e., distributions whose tails decay exponentially) the system is expected to be mostly in a periodic-like propagation regime, with the statistics of the period time being computable from the distribution of the fluctuations, as outlined above.

References

[1] N.F. Mott, Engineering 165 (1948) 16.
[2] A.A. Griffith, Phil. Trans. R. Soc. A 221 (1920) 163.
[3] A.N. Stroh, Adv. Phys. 6 (1957) 418; L.B. Freund, J. Mech. and Phys. of Solids I-20 (1972) 129; ibid. II-20 (1972) 141; ibid. III-21, (1973) 47; ibid. IV-22 (1974) 137.
[4] K. Ravi-Chandar, W.G. Knauss, Int. J. Fract. 25 (1984) 247 and references therein; ibid. 26 (1984) 65.
[5] W.B. Bradley, A.S. Kobayashi, Eng. Fract. Mech. 3 (1971) 317; R.C. Hoagland et al., Metallurgical Trans. 3 (1972) 123; L.B. Freund, in: F. Erdogan (Ed.), Mechanics of Fracture, AMD 19, ASME, NY, 1976; M.F. Kanninen, J. Mech. Phys. Sol. 25 (1977) 69; P.B. Crosely, E.J. Ripling ASTM-STP 627 (1977) 372.
[6] R. Thomson, Solid State Physics Vol. 39, p. 1.
[7] B.L. Holian, R. Blumenfeld, P. Gumbsch, Phys. Rev. Lett. 78 (1997) 78.
[8] A.K. Green, P.L. Pratt, Eng. Fract. Mech. 6 (1974) 71; W. Doll, J. Mat. Sci. 10 (1975) 935.
[9] J. Fineberg, S.P. Gross, M. Marder, H.L. Swinney, Phys. Rev. Lett. 67 (1991) 457; Phys. Rev. B 45 (1992) 5146; J. Dear, H. MacGillivray, in: H.P. Rosmanith, A.J. Rosakis (Eds.), Dynamic Failure of Materials, Elsevier, London, 1991; W.G. Knauss, P.D. Washabaugh, Int. J. Fract. 65 (1993) 189.
[10] M. Marder, Phys. Rev. Lett. 66 (1991) 2484; J.S. Langer, Phys. Rev. Lett. 70 (1993) 3592; D.A. Kurtze, D.C. Hong, ibid. 71 (1993) 847.
[11] E.H. Yoffe, Phil. Mag. 42 (1951) 739.
[12] L.B. Freund, in: W.G. Knauss, K. Ravi-Chandar, A.J. Rosakis (Eds.), Proceedings of the Workshop on Dynamic Fracture, CALTECH, (1983); C.C. Ma, L.B. Freund, Brown University Report, September 1984.
[13] R. Blumenfeld, Mesoscale fracture propagation: Noise-free and noise-driven steady states, periodicity and intermittency, Phys. Rev. E, submitted.

[14] B.V. Kostov, Appl. Math. Mech. (PMM) 30 (1966) 1241.
[15] E.g., J.W. Dally, W.L. Fourney, G.R. Irwin, Int. J. Fracture 159 (1985) and references therein.
[16] J.W. Dally, A. Shukla, Mech. Res. Comm. 6 (1979) 239; T. Kobayashi, J.T. Metcalf, in: Crack Arrest Methodology and Applications, ASTM-STP 711 (1980) 128; J.F. Kalthoff, in: Proceedings of the Workshop on Dynamic Fracture, CALTECH 11 (1983); K. Ravi-Chandar, W.G. Knauss, Int. J. Fracture 26 (1984) 141.
[17] J.R. Rice, Private communication.
[18] S.J. Zhou, P.S. Lomdahl, R. Thomson, B.L. Holian, Phys. Rev. Lett. 76 (1996) 2318.