

Basic Laws of Granular Materials

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Synopsis

Granular materials can exist in infinitely many configurations, but under well defined external influences and conditions can exhibit perfectly reproducible behaviour, and therefore must be possible to describe by statistical mechanical laws. These must be quite different then the traditional, Hamiltonian, statistical mechanics since the dynamics involved in changing from one configuration to another is dominated by friction. The ergodic, self-sustained equilibration of conventional statistical mechanics is then replaced by externally induced changes from one jammed configuration to another. Several questions arise, which we will attempt to answer here:

1. How does one specify a reproducible state of a granular system to which it can return after disturbance?
2. Stress due to external fields or boundary forces will propagate through the granular medium even if the grains are perfectly rigid and cannot be strained. This means that the stress is not related to the strain and stress-strain constitutive information is redundant. These are termed isostatic states. Even if the grains are not perfectly rigid the system can be isostatic. What are then the macroscopic constitutive equations required to determine the stress?
3. It appears that fluctuations of local properties are important to the stress equations. What exactly is the key quantity which fluctuates and how can the distribution of the fluctuations be found?
4. The three points above are theoretical in nature, but they must be supported experimentally. It is not difficult to generate spectacular but incomprehensible effects in granular materials, but there are experiments that really test basic fundamental concepts and these will be described.

§1 Introduction

Many granular and particulate systems have been studied in the literature and there is a wide range of parameters and physical states that they support [1]. Here we confine ourselves to jammed ensembles of perfectly hard particles. There are extensive studies in the literature of suspensions of particles in liquids or gases

using various methods, including Stokes or Einstein fluid mechanics and Boltzmann or Enskog gas mechanics. These, however, are not jammed and we therefore discuss them no further. This chapter is not intended as a comprehensive review but rather as an interim report on the work that has been done by us to date.

The simplest material for a general jammed system is that of hard and rough particles, ideally perfectly hard and infinitely rough. To a lesser extent it is also useful to study perfectly hard but smooth particles. The former is easily available in nature, for example sand, salt, etc, and we prefer to focus on this case. Nevertheless, the discussion can be readily extended to systems of particles of finite rigidity, as has been shown recently [2]. In jammed systems particles touch their neighbours at points, which have to be either predicted or observed. At these contact points the particles exert on one another forces that must obey Newton's laws. In general, determination of the structure and the forces requires prior knowledge about the history of formation of the jammed system. For example, if grains of sand are poured from a narrow orifice onto a plane they will form a conical sand pile which is known to have a minimum of pressure under the apex [3]. If, however, the sand grains are poured uniformly into a right cylinder standing on a plane the cylinder will fill at approximately a uniform rate, producing a relatively flat surface and a uniform pressure on the plane. If one starts pouring the sand from a narrow orifice into a cylinder and changes to a uniform source when the edges of the pile reaches the cylinder walls then the original sand pile will be buried eventually by the uniform deposition and the pressure on the plane is some mixture of the two earlier pictures. Therefore, just given a cylinder full up to a certain level by sand is *insufficient* to determine the pressure at the bottom. Without knowledge of the formation history only a detailed tomography of the individual grains can help the investigator. This is usually the situation in the systems relevant to Soil Mechanics and to Civil Engineering.

But there is another situation which brings the problem into the realm of physics. In this set-up the cylinder of sand is prepared in such a way that there is an analogue of equilibrium statistical mechanics which opens the door to ab initio calculations of configurations and forces. Suppose the cylinder of sand is shaken with an amplitude A and a frequency ω , each shake being sufficient to break the jamming conditions and reinstate the grains for the next shake. The sand will then occupy a volume V which is a function of A and ω , $V(A, \omega)$. Changing A to A' and ω to ω' one will get a new volume $V' = V(A', \omega')$. If we now return to A and ω we will again find that the volume is $V(A, \omega)$. This suggests that, in analogy with the microcanonical ensemble in thermodynamics, the sand will possess an entropy which is the logarithm of the number of ways the N grains of sand will fit into the volume V , namely, the conventional expression for the entropy

$$S(E, V, N) = \log \int \delta(E - \mathcal{H}) d\{\text{all degrees of freedom}\} , \quad (1)$$

is replaced by

$$S(V, N) = \log \int \delta(V - \mathcal{W}) \Theta d\{\text{all degrees of freedom}\} , \quad (2)$$

where \mathcal{W} is a function of the structural characteristics of the grains that gives the

volume for any arbitrary configuration of grains and Θ is the condition that all grains are touching their neighbours in such a way that the system is in mechanical equilibrium. If eq. (2) is accepted (its derivation is given below) then one can pass to the canonical ensemble replacing the conventional expressions on the left by those on the right;

$$T = \frac{\partial E}{\partial S} \quad \leftrightarrow \quad X = \frac{\partial V}{\partial S} , \quad (3)$$

$$F = E - TS \quad \leftrightarrow \quad Y = V - XS . \quad (4)$$

In these, X is named the compactivity of the system, since $X = 0$ corresponds to maximum density and $X = \infty$ is where the condition of mechanical equilibrium fails due to a topology that cannot support the intergranular forces.

Detailed studies of the density of shaken granular systems as a function of the number of ‘tappings’ and the force of a tap were first given by the Chicago group [4] and fit in with the above theoretical arguments.

§2 Statistical Mechanics

Consider a cylinder containing granular material whose base is a diaphragm that can oscillate with frequency ω and amplitude A . Suppose one vibrates the system for a long time. When the vibration is turned off the granular material occupies a volume $V_0 = V(A, \omega)$. Repeating the process with ω_1 and A_1 gives a volume $V_1 = V_1(A_1, \omega_1)$. Returning now to ω and A , it has been found that the system returns to $V(A, \omega)$. This is surely what one would expect, nevertheless the experiment, done firstly by the Chicago group [4], is new. A different version of this experiment has been carried out in our department [5][6]: powdered graphite, after first being assembled, has a low density, as found by measuring its conductivity. But as it is shaken and allowed to come to rest again it exhibits a higher conductivity. Upon cycling the load applied to the powder one reaches, and moves along, the reversible curve shown in figure 1. By using a simple effective medium approximation [7] it is possible to estimate the mean coordination number as a function of the conductivity using the relation [6][8]. We shall see later that $\langle z \rangle$ is a parameter that plays an central role in the behaviour of granular materials.

The first rigorous theory of statistical mechanics came when Boltzmann derived his equation and proved that it describes a system whose entropy increases until equilibrium is achieved with the Boltzmann distribution. He needed a physical specification, that of a low density gas where he could assume only two body collisions, and a hypothesis, the Stosszahlansatz, that memory of a collision was not passed from one collision to another. The question is can we do the same for a powder?

Assuming that the grains are incompressible, a physical condition is that all grains are immobile when an infinitesimal test force is applied to a grain, namely, there are no ‘rattlers’ which carry no stress at all. A system is jammed when all grains have enough contacts and friction is such that there is a finite threshold that a force has to exceed for motion to initiate. The hypothesis we need is that when the external force, say from a diaphragm, propagates stress through the system, then for a particular A and ω there exist bounded regions where motion results which

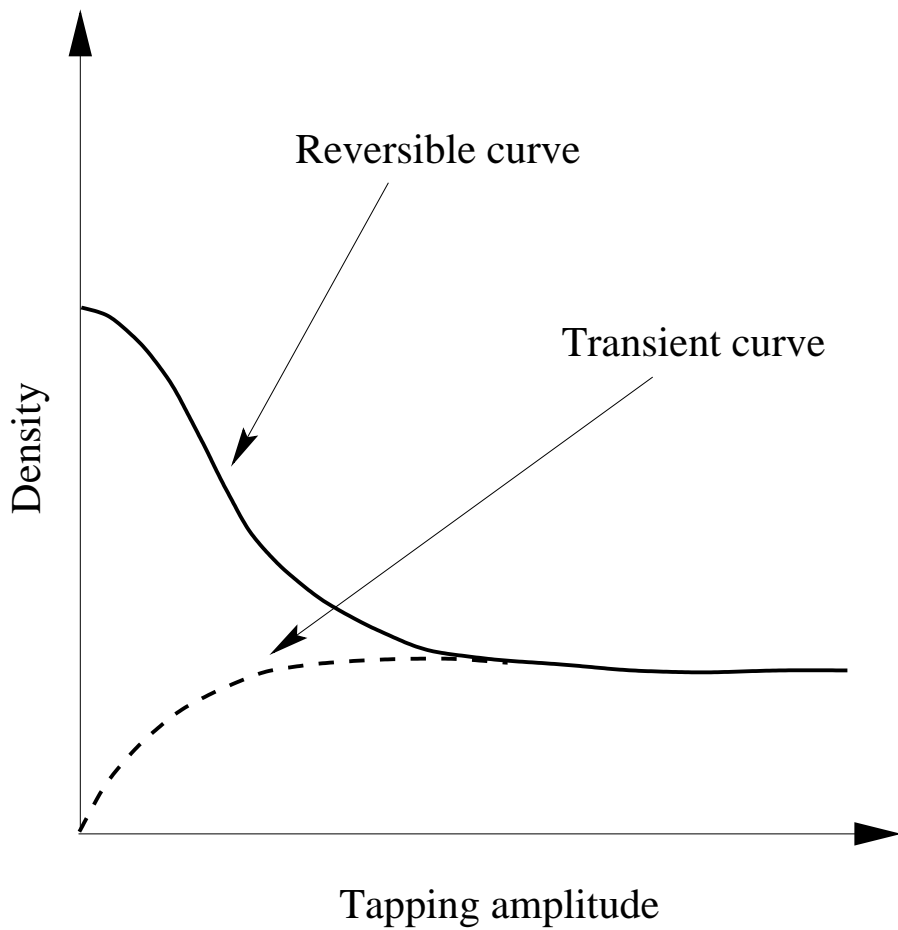


Figure 1: A sketch of the density of granular matter in a vessel after being shaken at amplitude A . Adapted from [4]

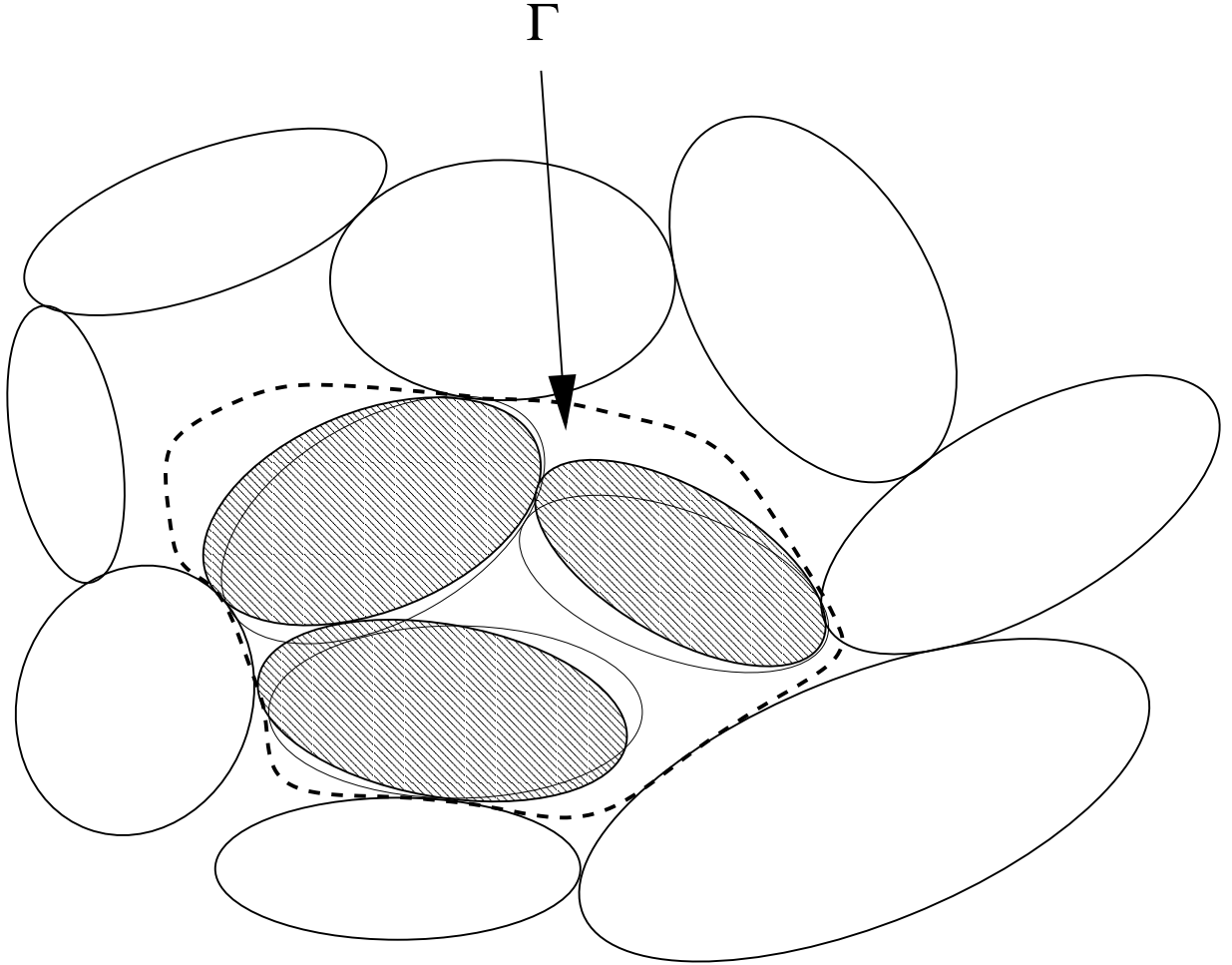


Figure 2: An example of two states of a granular system that differ only by the positions of three particles confined to within a region, Γ

rearranges the grains. We assume that outside these regions no rearrangement takes place. An example is illustrated in figure 2, where the region Γ consists of three particles that can rearrange in several configurations, of which two are sketched. Given the equation characterising the boundary of Γ and the configuration of the grains inside it, there must exist a function \mathcal{W}_Γ that gives the volume of Γ in terms of variables which describe the local geometric structure and the boundary grains. Since the system is shaken reversibly then under the shake \mathcal{W}_Γ remains the same $\mathcal{W}_\Gamma = \mathcal{W}_{\Gamma'}$ and for the entire system

$$\sum \mathcal{W}_\Gamma = \sum \mathcal{W}_{\Gamma'} . \quad (5)$$

We can now construct a Boltzmann equation. There must be a probability f of finding any configuration with a specification of positions and orientations. Under a shake

$$\frac{d\mathcal{P}}{dt} = \int K(\Gamma, \Gamma') (\Pi_\Gamma f^\Gamma - \Pi_{\Gamma'} f^{\Gamma'}) \quad (6)$$

where \mathcal{P} consists of the probabilities f^Γ of finding particular configurations of grains

inside regions Γ and their boundary specifications. The kernel function K contains all the information on the contacts between grains and the constraints on the forces expressed via δ -functions.

Now we are at the same situation as Boltzmann, for the steady state will depend only on $\delta(\mathcal{W}_\Gamma - \mathcal{W}_{\Gamma'})$ and the jamming specification. This is the analogue of the conservation of kinetic energy of two particles under collision in conventional statistical mechanics. Eq. (2) means that the probability f which satisfies (2) is

$$f^\Gamma = e^{Y/X - \mathcal{W}_\Gamma/X} \Theta \quad (7)$$

where Θ specifies the jamming conditions and $e^{Y/X}$ is the normalisation. We can go further and deduce the entropy of the powder by

$$S = - \int f \log f , \quad (8)$$

where we have dropped, for convenience, the indices Γ and Γ' . From (4) we can derive, using symmetry arguments in the same way that Boltzmann did,

$$\frac{dS}{dt} = \int K \Pi f \left(\frac{\Pi f}{\Pi f'} - 1 \right) \log \frac{\Pi f}{\Pi f'} . \quad (9)$$

Since K and Πf are positive definite, as is $(x - 1) \log x$ for $x > 0$, then

$$\frac{dS}{dt} > 0 \quad \text{until} \quad f = e^{(Y - \mathcal{W}_\Gamma)/X} . \quad (10)$$

The Boltzmann approach leads naturally to the canonical ensemble, but the result (4) was first put forward for the micro-canonical ensemble [9]

$$S = \log \int \delta(V - \mathcal{W}) \Theta d\{\text{all degrees of freedom}\} \quad (11)$$

where now \mathcal{W} is the complete volume function and Θ the complete jamming condition. This form is the analogue of

$$S = \log \int \delta(E - \mathcal{H}) d\{\text{all degrees of freedom}\}$$

and the usual result

$$F = E - TS$$

becomes

$$Y = V - XS . \quad (12)$$

Similarly, the analogue of the temperature $T = \partial E / \partial S$ is now the compactivity

$$X = \frac{\partial V}{\partial S} . \quad (13)$$

This discussion, which has been presented for perfectly hard grains, can be readily extended to the analysis of grains that have internal energy. This leads to

$$S = \int \int \delta(E - \mathcal{H})\delta(V - \mathcal{W})\Theta d\{\text{all degrees of freedom}\} \quad (14)$$

and we obtain

$$e^{S-E(\partial S/\partial E)_{V,N}-V(\partial S/\partial V)_{E,N}} = \int e^{-\mathcal{H}(\partial S/\partial E)-\mathcal{W}(\partial S/\partial V)}\Theta d\{\text{all degrees of freedom}\} \quad (15)$$

or

$$e^{S-E/T-V/X} = \int e^{-\mathcal{H}/T-\mathcal{W}/X}\Theta d\{\text{all degrees of freedom}\} . \quad (16)$$

The Gibbs relation

$$S - E \left(\frac{\partial S}{\partial E} \right)_{V,N} - V \left(\frac{\partial S}{\partial V} \right)_{E,N} = S - E/T - PV = -G \quad (17)$$

identifies the inverse of the compactivity as

$$\frac{1}{X} = \left(\frac{\partial E}{\partial V} \right)_{S,N} \left(\frac{\partial S}{\partial E} \right)_{V,N} = -\frac{P}{T} \quad \text{as } T \rightarrow 0 . \quad (18)$$

We regard this relation, however, as a curious formal analogue rather than a useful formula. Although in general, entropies due to internal thermal effects and configurational rearrangements mix, the two can be readily separated (i.e. a heap of hot sand will have many of the characteristics of a heap of cold sand) and we can write

$$S = S_{th} + S_{conf} . \quad (19)$$

It is interesting to note that confirmation of this ‘thermodynamics’ of granular systems by numerical simulations has used the mixed, rather than the purely configurational, approach [10].

One can go further to the Grand canonical ensemble

$$\Omega = S - E \left(\frac{\partial S}{\partial E} \right)_{V,N} - V \left(\frac{\partial S}{\partial V} \right)_{E,N} - N \left(\frac{\partial S}{\partial N} \right)_{E,V} = S - E/T - V/X - N\mu/T . \quad (20)$$

Since there can be many different kinds of grains then the last term should really be a sum over N_i and μ_i , but we have not looked into such systems yet.

If the system is subject to an external stress on its surface, P_{ij} then one can be even more general and notice that S becomes $S(V, N, P_{ij})$ and (now discarding E and keeping N fixed)

$$\Omega = S - V \left(\frac{\partial S}{\partial V} \right)_{P_{ij}} - P_{ij} \left(\frac{\partial S}{\partial P_{ij}} \right)_V \quad (21)$$

leading to a distribution

$$e^{-S+(V-\mathcal{W})\frac{\partial S}{\partial V}+(P_{ij}-\Pi_{ij})\frac{\partial S}{\partial P_{ij}}} , \quad (22)$$

where the simplest case only involves the external pressure P_{kk} , and Π_{kk} is related to the total force moment $\sum_{grains} f_i r_i / V_{grain}$. This latter form is briefly discussed below. Having named $\frac{\partial S}{\partial V}$ the compactivity, we name the quantity $\frac{\partial S}{\partial p}$, where p is the scalar pressure, *angoricity*. Note that in general the angoricity is the analogue of a *tensorial temperature*, $\frac{\partial S}{\partial P_{ij}}$.

Formula (11) was presented many years ago [9] but did not find wide acceptance. This was partly due to a lingering skepticism and partly due to the inexistence of an exact way to characterise the analogue of a Hamiltonian, the volume function \mathcal{W} . Both these problems have been resolved. First, numerical simulations have appeared that validated the formalism [11]. The second development involved the discovery of an exact volume function both in two dimensions [12][13] and in three dimensions [14]. Nevertheless, to our minds, the validity of this approach was already implicit in the experiment in reference [4].

§3 Volume functions and Forces in Granular Systems

We have seen in §2 that, provided a mechanism for changing configurations can be found, such as tapping and vibrational agitation, a reversible curve can be achieved. This implies that a statistical mechanical approach can be applied to this set of states in powders and that the probability distribution is governed by

$$e^{(Y-\mathcal{W})/X} \Theta . \quad (23)$$

This is already enough for a simple theory of miscibility [9] and indeed any application of the conventional thermodynamic function $\exp(-(F - \mathcal{H})/k_B T)$ will have an analogue for granular systems. However, these systems also enjoy several new problems that have no equivalent in conventional thermal systems. One such a problem concerns the distribution of forces and stresses within the granular packing. Many of the most interesting issues concerning force transmission in, e.g. heaps of particles, lie outside the above framework, for the force exerted by a sand pile on its base depends sensitively on how it was created. Nevertheless, there are quite a few splendid problems that can be tackled with the analytical tools we have already.

The simplest case is probably that of perfectly hard and rough particles ('perfect' must be understood to not fully apply when the material is assembled, but once it has consolidated we can restrict ourselves to the application of forces below the yield limit). In the following we consider particles of arbitrary shapes and sizes. Presuming that the material is in mechanical equilibrium, force and torque balance must be satisfied. Let us consider a part of the material sketched in figure 3. We assume for simplicity that no two neighbouring particles contact at more than one point. This assumption is not essential to our discussion but it leads, as we shall see in the following, to the conclusion that in two dimensions the material is in isostatic mechanical equilibrium when the average coordination number per grain is exactly three. Figure 3 shows a particular grain g in contact with three neighbours, g' , g'' and g''' . The contact point between, say, grains g and g' is $\bar{\rho}^{gg'}$ and each grain is assigned a centroid,

$$\bar{\rho}^g = \frac{1}{z_g} \sum_{g'} \bar{\rho}^{gg'} , \quad (24)$$

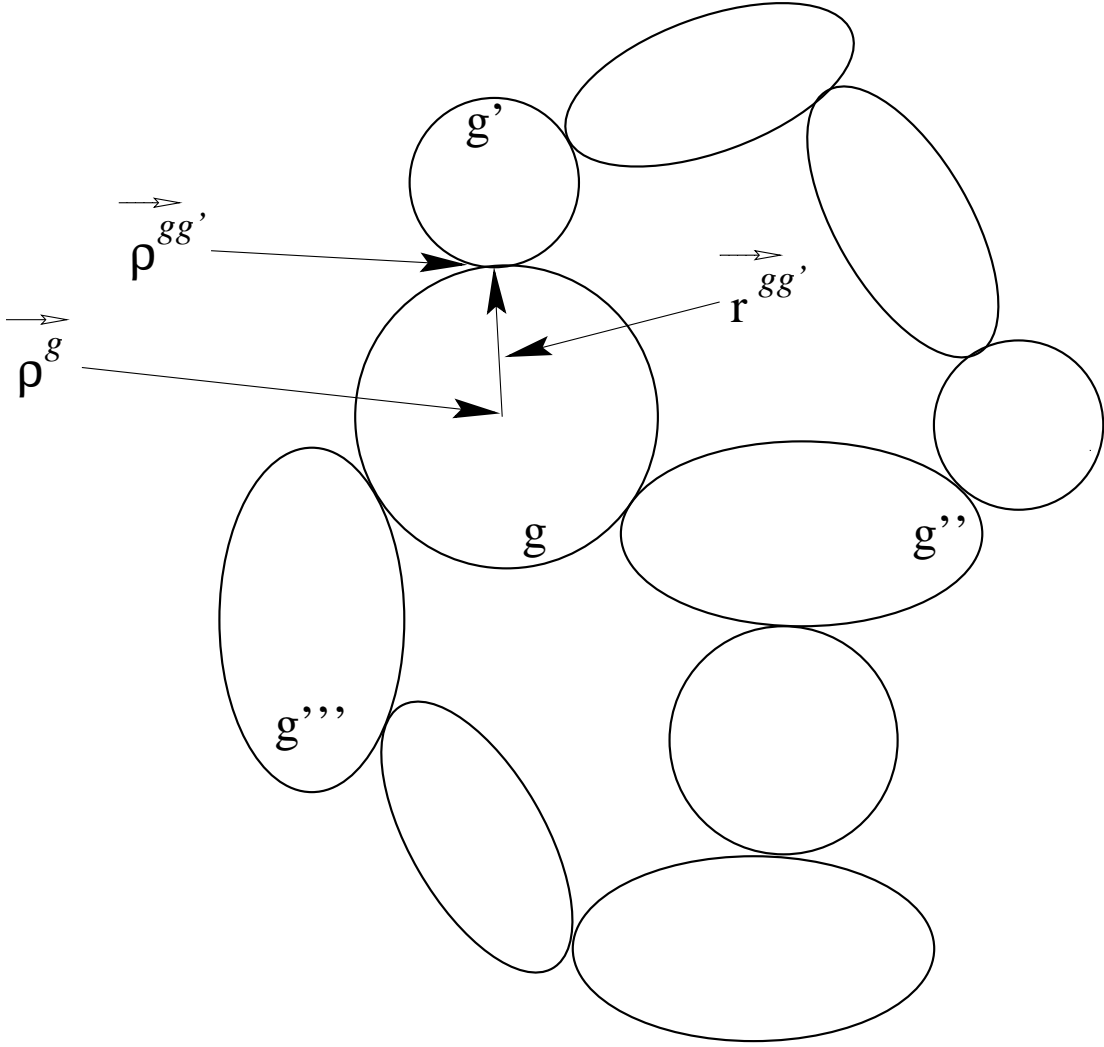


Figure 3: A particle g in contact with three neighbours g' , g'' and g''' . $\vec{\rho}^{gg'}$ is the position vector of the contact between g and g' ; $\vec{\rho}^g$ is the centroid of the contact points; $\vec{r}^{gg'}$ points from the centroid to the contact point between g and g' ; $\vec{R}^{gg'} = \vec{r}^{gg'} - \vec{r}^{g'g} = -\vec{R}^{g'g}$; $\vec{S}^{gg'} = \vec{r}^{gg'} + \vec{r}^{g'g} = \vec{S}^{g'g}$.

that is defined to be the mean of the positions of all its z_g contacts. The vector

$$\vec{r}^{gg'} = \vec{\rho}^{gg'} - \vec{\rho}^g \quad (25)$$

points from the centroid of grain g to the point of its contact with grain g' . The grains g and g' also exert a force on one another through the contact and let $\vec{f}^{gg'}$ be the force that g exerts on g' . For later use we also define the vectors

$$\vec{R}^{gg'} = \vec{r}^{gg'} - \vec{r}^{g'g} = -\vec{R}^{g'g} \quad (26)$$

and

$$\vec{S}^{gg'} = \vec{r}^{gg'} + \vec{r}^{g'g} = \vec{S}^{g'g} . \quad (27)$$

Balance of forces and torque moments gives

$$\sum_{g'} \vec{f}^{gg'} = \vec{G}^g \quad (28)$$

$$\sum_{g'} \vec{f}^{gg'} \times \vec{r}^{gg'} = 0 , \quad (29)$$

where \vec{G}^g is the external force acting on grain g . Newton's third law requires that at each contact

$$\vec{f}^{gg'} + \vec{f}^{g'g} = 0 . \quad (30)$$

Various useful tensors can be generated using these vectors:

$$\begin{aligned} \hat{\mathcal{E}}_{ij}^g &= \sum_{g'} R_i^{gg'} R_j^{gg'} \\ \hat{\mathcal{F}}_{ij}^g &= \sum_{g'} f_i^{gg'} f_j^{gg'} \\ \hat{\sigma}_{ij}^g &= \frac{1}{2} \sum_{g'} (f_i^{gg'} r_j^{gg'} + f_j^{gg'} r_i^{gg'}) . \end{aligned} \quad (31)$$

The latter is sometimes known as the Love stress tensor. Other 'fabric tensors' that have appeared already in the literature, can also be defined from these quantities, e.g. $\sum_{g'} r_i^{gg'} r_j^{gg'}$. We will show first that a simple theory of granular systems can be expressed in terms of these tensors. However, it does not yield a complete description. A new geometric characterisation has been formulated, which makes it possible to construct an exact microscopic theory of two-dimensional systems, and this will be described below and in section §4.

In three dimensions the 3×3 tensor $\hat{\mathcal{E}}_{ij}^g$ has three Euler angles of orientation and three eigenvalues, λ_1^2 , λ_2^2 , λ_3^2 , whose combinations have direct physical interpretations:

$$\langle \sum_i \hat{\mathcal{E}}_{ii} \rangle = \langle \lambda_i^2 \rangle = 3 \times (\text{the average radius squared}) \quad (32)$$

$$\langle \lambda_1^2 \lambda_2^2 \lambda_3^2 \sum_i \lambda_i^{-2} \rangle = 3 \times (\text{the average cross section}) \quad (33)$$

and

$$\langle \lambda_1^2 \lambda_2^2 \lambda_3^2 \rangle = \{\text{the average volume squared}\} \quad (34)$$

The total volume is approximately

$$V \approx \frac{1}{2} \sum_g \sqrt{\det(\hat{\mathcal{E}}^g)} . \quad (35)$$

Thus, from $\hat{\mathcal{E}}^g$ we can produce a first approximation to the volume function \mathcal{W} of the entire system

$$\mathcal{W} = \sum_g \mathcal{W}^g = \frac{1}{2} \sum_g \sqrt{\det(\hat{\mathcal{E}}^g)} . \quad (36)$$

More recently Ball and Blumenfeld [12] have found an exact form for \mathcal{W} in two dimensions, using a new geometric tensor that characterises differently the local microstructure around grains. This geometric tensor is constructed as follows. For lack of sufficient symbols we shall use in what follows \vec{R} and \vec{r} again but these should not be confused with the quantities defined in equations (25) and (26). First, connect all the contact points around grain g by vectors \vec{r}^{gl} that circulate clockwise, as shown in figure 4. The choice of this direction is not essential but it is important that these vectors circulate in the same direction around *all* grains. The vectors \vec{r}^{gl} form a network that spans the system which we term the *contact network*. In two-dimensions the grains form closed loops that enclose voids and around these loops the vectors \vec{r}^{gl} circulate in the anti-clockwise direction. Each \vec{r}^{gl} is uniquely identified by the grain g that it belongs to and the void loop l that it circulates. Next, define the centroid of loop l as the mean position vector of all the contact points around it

$$\vec{\rho}^l = \frac{1}{z_l} \sum_{g, g' \in \partial l} \vec{\rho}^{gg'} , \quad (37)$$

where z_l is the number of grains around the loop and the sum is over the grains that surround it, ∂l . Finally, define a vector, \vec{R}^{gl} , that extends from the centroid of grain g to the centroid of void loop l (see figure 5),

$$\vec{R}^{gl} = \vec{\rho}^l - \vec{\rho}^g . \quad (38)$$

The vectors \vec{R}^{gl} also form a network that spans the system and this network is the dual of the contact network. The Ball-Blumenfeld basic geometric tensor is expressed in terms of the outer product of these vectors

$$\hat{C}_{ij}^g = \sum_l r_i^{gl} R_j^{gl} , \quad (39)$$

where i, j stand for x, y and the sum runs over all the loops that surround grain g . The anti-symmetric part of each of the terms in the sum (39) can be written as

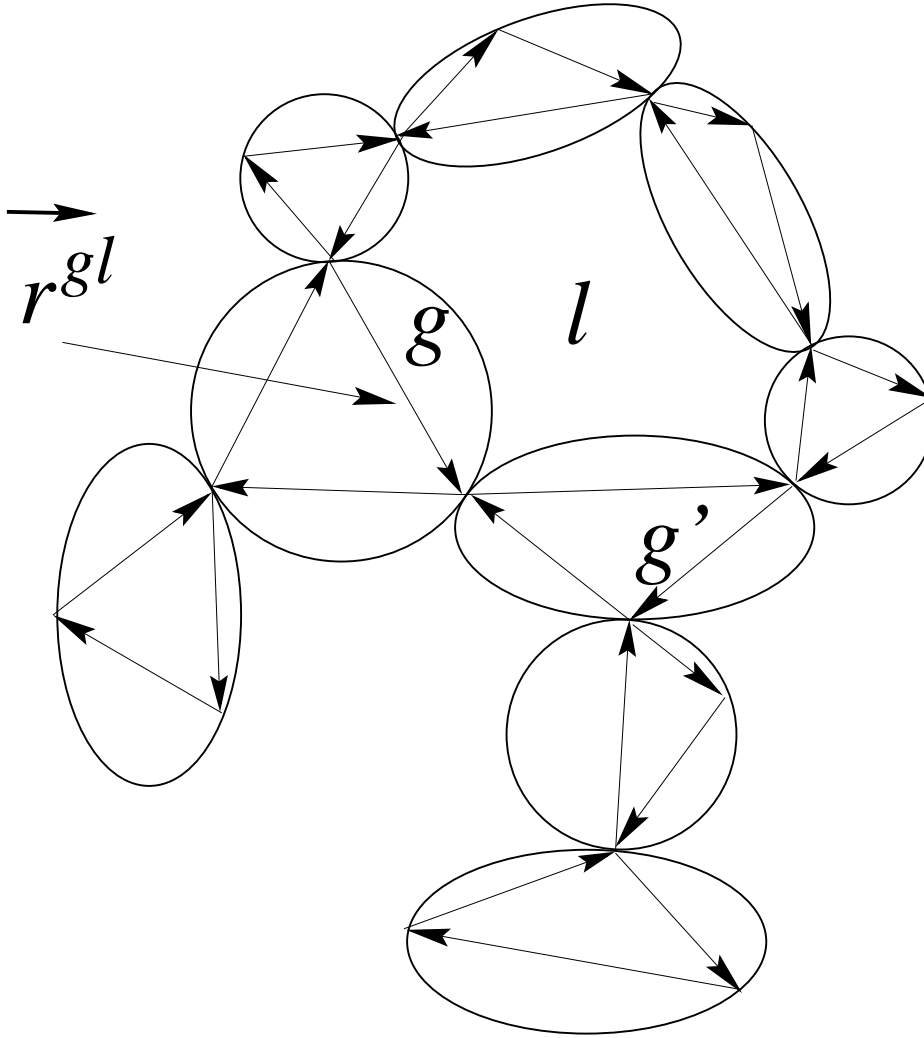


Figure 4: The circulation of loops of \vec{r} -vectors around grains, e.g. g , is in the clockwise direction and around voids, e.g. l , in the anti-clockwise direction.

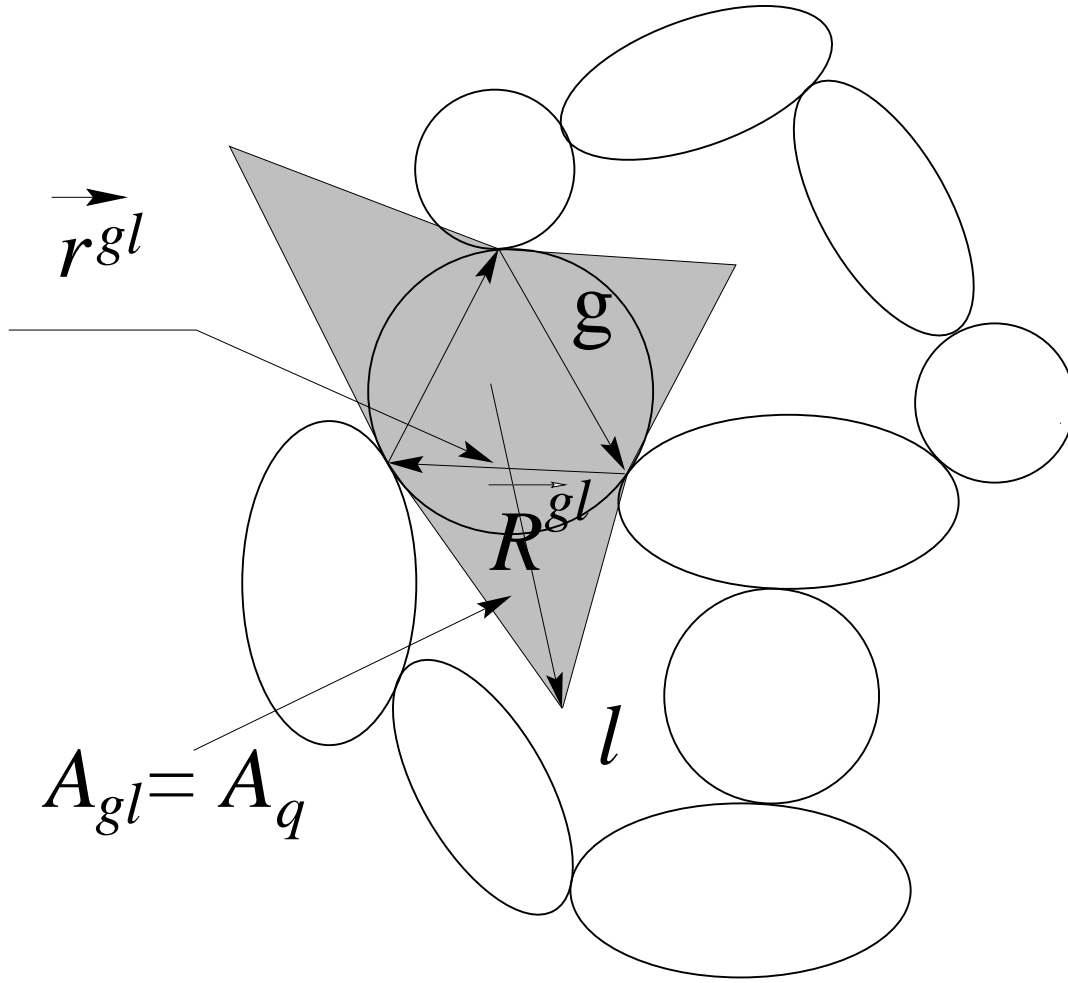


Figure 5: The geometric variables around grain g . The \vec{r} -network connects the contact points around grains; for example, \vec{r}^{gl} is a vector connecting two neighbouring contact points around grain g which are on the boundary of void loop l . The vector \vec{R}^{gl} extends from the centroid of grain g to the centroid of void loop l . The pair $\vec{r}^{gl} - \vec{R}^{gl}$ forms a quadrilateral q that is the elementary unit of the structure - the Quadron. The area of the quadron is termed $A_{gl} = A_q$.

$$\mathcal{A}(\vec{r}^{gl} \vec{R}^{gl}) \equiv A_{gl} \hat{\epsilon} , \quad (40)$$

where $\hat{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the unit antisymmetric tensor corresponding to $\frac{\pi}{2}$ -rotation in the plane. The prefactor A_{gl} is exactly the area of the quadrilateral of which the vectors \vec{r}^{gl} and \vec{R}^{gl} are the diagonals (see figure 5). A key observation is that the areas of the quadrilaterals tile the entire system without holes and with no overlaps. By summing these areas over the quadrilaterals that surround grain g we obtain the area associated with this grain

$$A_g = \sum_l A_{gl} . \quad (41)$$

A summation of this quantity over all grains gives the area of the entire two-dimensional system. Thus, the volume function in two dimensions is exactly

$$\mathcal{W} = \sum_g A_g = \sum_l A_{gl} . \quad (42)$$

Note that we could index each quadrilateral by q and sum over all q directly instead of over the grains g and the void loops l . This indicates that the basic building blocks of the system are not the grains, as one would initially expect. Rather, each grain can be regarded as composed of z_g internal elements, the quadrilaterals, and these are the fundamental quasi-particles (or excitations, in the language of conventional statistical mechanics) of the system. In two-dimensional isostatic, or marginally rigid, systems have on average three quadrilaterals per grain and we term these elementary quasi-particles ‘Quadrons’.

To make use of this identification it is necessary to determine the distribution of volumes A_q in any given system. This information, combined with the behaviour of the density of states Θ (which, as in conventional thermodynamic system, is expected generically to vary as a power law), will make it possible to deduce the compactivity of the system X by fitting it to an exponential form. Alternatively, it makes it possible to estimate the density of states analytically and proceed to calculate the partition function

$$Z = \int \dots \int e^{-\beta \sum_{q=1}^{3N} A_q} \Theta(\{A_q\}) dA_q \quad (43)$$

as a function of the compactivity.

The volume function (41) also makes it possible to identify a compact phase space of degrees of freedom, the vectors $\vec{r}^{gl} = \vec{r}^q$. There are altogether $3N$ such vectors, on average three per grain. These, however, are not all uncorrelated due to the constraints imposed by the topology of the structure. Basically, we need to determine how many *independent* degrees of freedom there are. A very significant advantage of the exact volume function (41) is that it enables to pinpoint the correlations amongst these vectors [14]. The key to this lies in the observation that the topological constraints that give rise to the correlations originate from the *irreducible loops* in the structure. The irreducible loops are the fundamental loops of which all other loops can be composed, as shown in figure 6. There are two types

of irreducible loops: grain loops, which consist of the vectors \vec{r}^q connecting the contacts around individual grains and void loops, which consist of vectors \vec{r}^q circulating around individual voids. There are N of the former, one per grain, and there are M of the latter, giving altogether $N + M$ dependent vectors. To determine the number of voids M we can employ Euler's relation [15] on the relations between edges, cells and vertices, combined with the fact that the mean coordination number in two dimensions is exactly three per grain for these systems. This gives that there are on average six grains around a void. This means that there are two grains per void loop and therefore that $M = N/2$. Thus, $N + M = 3N/2$ and of the $3N$ vectors \vec{r}^q only half are independent. With two degrees of freedom per vector, this gives that the phase space is $3N$ -dimensional. Turning attention back to the quadrons, this argument leads to a surprising coincidence: the number of independent degrees of freedom is the same as the number of quadrons! This suggests that in two dimensions one can get rid of the function $\Theta(\{A_q\})$ in the partition function (43) by integrating over quadrons rather than the independent vectors.

$$Z = \int P(\{A_q\}) \prod_{q=1}^{3N} e^{-\beta A_q} dA_q, \quad (44)$$

where $P(\{A_q\})$ is the correlated probability density of the quadron areas. Various conventional statistical mechanical methods can be used to evaluate this probability density (e.g., cluster expansion). The simplest approximation would be to assume that the quadrons are independent, namely,

$$P(\{A_q\}) = \prod_{q=1}^{3N} P_q(A_q), \quad (45)$$

where $P_q(A_q)$ is the probability density of the area of the q quadron. This form, which resembles the treatment of the density of states in conventional statistical mechanics, makes it simpler both to evaluate the partition function and to appreciate the implicit approximations that have been used in the literature when the volumes of the grains, rather than the quadrons, were taken as the fundamental particles of the system.

At present we know of no first-principles theory that gives the form of $P_q(A_q)$ for any system. Therefore, to make progress, we are required to make assumptions on this form. One simple approximation is to assume that the area of any quadron is chosen from a uniform distribution of average A_0 that lies between a maximum value $A_0 + \Delta$ and a minimal value $A_0 - \Delta > 0$

$$P(B_q) = \begin{cases} \frac{1}{2\Delta} & \text{if } A_0 - \Delta < B_q < A_0 + \Delta; \\ 0 & \text{otherwise.} \end{cases} \quad (46)$$

In this case the partition function is straightforward to compute;

$$Z = \left(\frac{e^{-2\beta A_0} \sinh(\beta \Delta)}{\beta \Delta} \right)^{3N} \quad (47)$$

and the mean total area of the system and its mean fluctuations are

$$\langle A_{system} \rangle = \frac{3N}{2} \left(2A_0 + \frac{1}{\beta} - \Delta \tanh(\beta \Delta) \right)$$

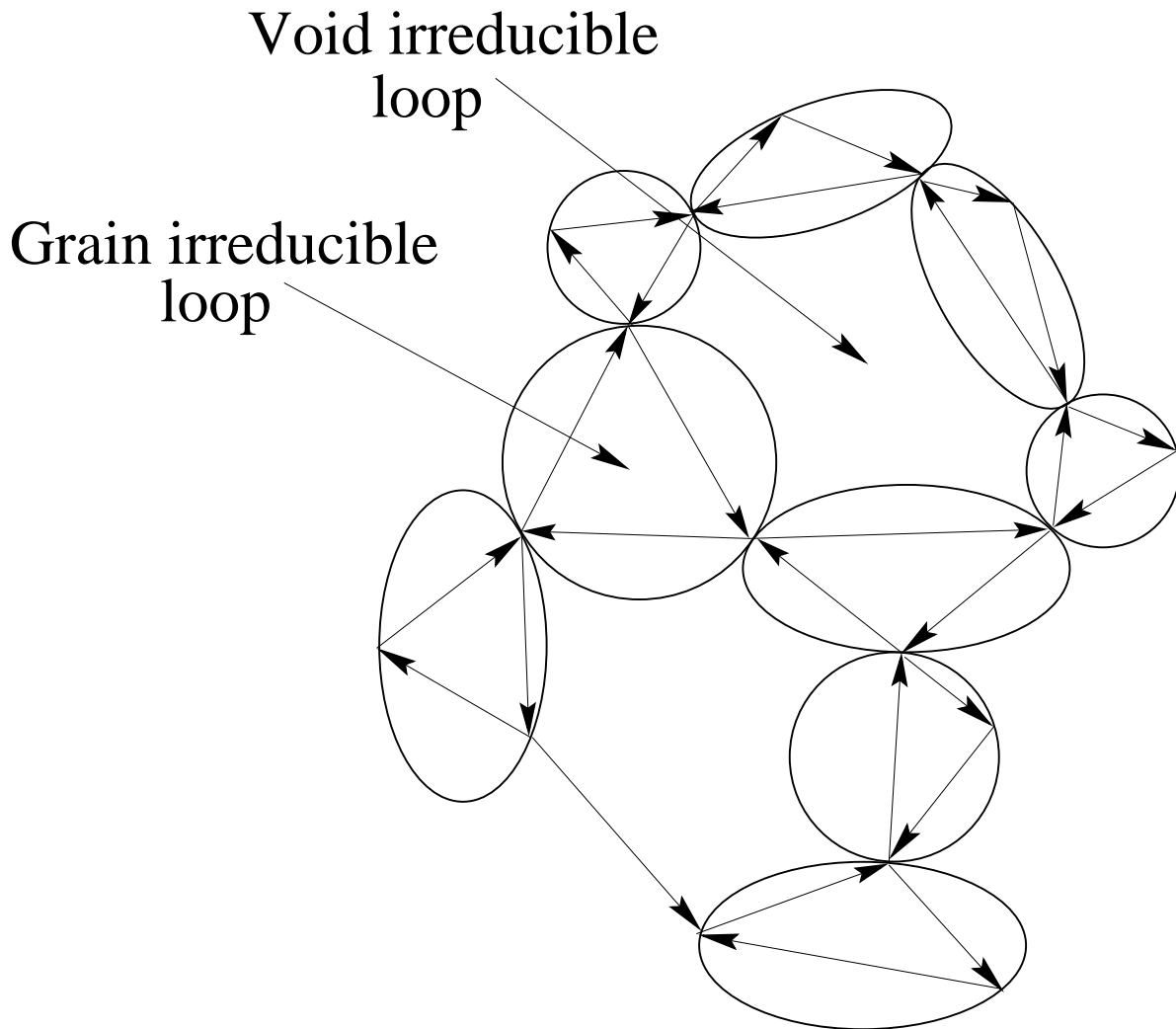


Figure 6: The irreducible loops consist of two types: loops circulating grains (in the clockwise direction) and loops circulating voids (in the anti-clockwise direction). All other loops in the networks can be decomposed into combinations of the irreducible loops.

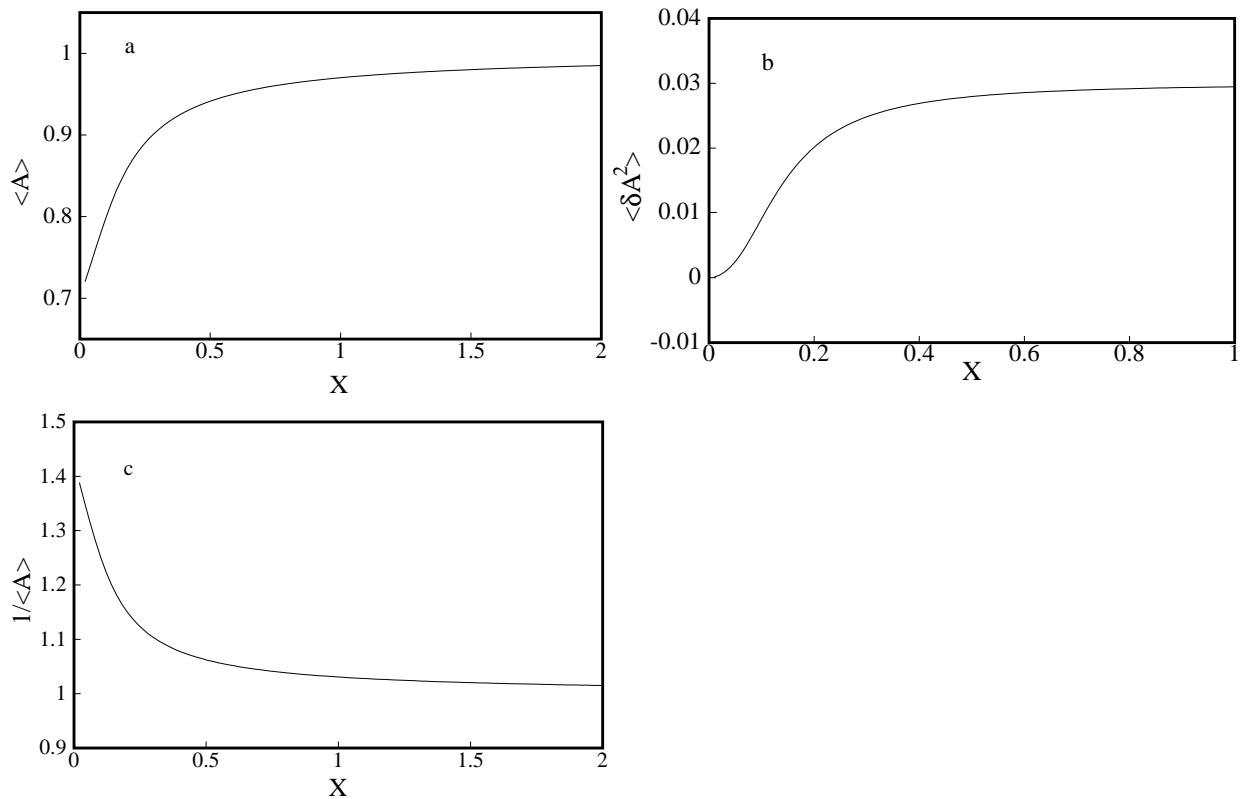


Figure 7: (a) The mean volume per quadron as a function of the compactivity for the density of states (the distribution of quadron areas) modelled by (46); (b) the mean volume fluctuations per quadron as a function of compactivity; (c) the mean density per quadron as a function of compactivity. The plots agree reasonably well with the experimental results of reference [4].

$$\langle \delta A_{system}^2 \rangle = \frac{3N}{2} \left(\frac{1}{\beta^2} - \frac{\Delta^2}{\sinh(\beta\Delta)^2} \right) . \quad (48)$$

$$(49)$$

A plot of the mean area and the fluctuations as a function of the compactivity X are shown in figure 7. Recalling that the compactivity increases monotonically with the external agitation of the system (the tapping in the Chicago experiment), one can invert the plot of the mean volume into a plot of the mean density (see figure 7c). The plot is in a reasonable agreement with the experimental results of reference [4].

§4 The Stress Field

As mentioned already, in a packing of perfectly hard grains there cannot be any strain involved in the determination of the stress. This observation extends to systems of grains that are not infinitely rigid [2], as long as the condition that the mean coordination number is three in two dimensions and four in three dimensions. The system is then isostatic or statically determinate. The conditions for frameworks to be statically determinate have been worked out already by Maxwell [16] and Levy [17]. Since the forces in isostatic states are statically determinable then stress-strain relations are *redundant* and therefore the equations for the stress field have to be

closed by another type of constitutive information. The new set of equations and their analysis is called isostaticity theory. In the following we shall make use the symbol $\hat{\sigma}$ for the stress tensor and σ_{ij} for its components. These should not be confused with the scalar conductivity σ in section §4. The global balance equations for the stress field in isostatic states are

$$\begin{aligned}\frac{\partial \sigma_{ij}}{\partial x_i} &= g_j \quad (\text{force balance}) \\ \sigma_{ij} &= \sigma_{ji} \quad (\text{torque balance})\end{aligned}\tag{50}$$

are the force and torque balance equations for $d(d+1)/2$ variables in d dimensions. The stress tensor has d^2 components to be determined and therefore further $d(d-1)/2$ equations are missing, one in 2D and three in 3D. These equations must depend on the geometry of the contacts and we derive them below. We shall give below two derivations. The first, after Edwards and Grinev [18], applies only to systems on the reversible curve. Systems not on this curve, such as sand piles, depend on the history of preparation. In sand piles the missing equations were postulated using two main hypotheses, ‘Fixed principal axis’ and ‘Oriented stress linearity’, each resulting in a different equation [19]. The second derivation presented here, after Ball and Blumenfeld [12], applies to generally quenched systems.

First approach:

Although the stress is a macroscopic variable it can be defined on the level of one grain by using the force moment, which on averaging over the volume becomes the stress

$$S_{ij}^g = \frac{1}{2} \sum_{g'} (f_i^{gg'} r_j^{gg'} + f_j^{g'g} r_i^{g'g})\tag{51}$$

where Newton’s equations in §2 apply. We need the probability $\mathcal{P}(\hat{\sigma}^g)$ and, more usefully, its Fourier transform

$$\mathcal{P}(\zeta^g) = \int e^{i \sum_{ij} \zeta_{ij}^g \sigma_{ij}^g} \Pi_{ij} d\sigma_{ij}^g.\tag{52}$$

Parameterising Newton’s equations Fourier analysis we are led to [18].

$$\mathcal{P}(\hat{\sigma}) = \int d\zeta \int d\eta e^{i \sum \eta^g F_{ext}^g} \Pi_{gg'} \delta(\zeta^g r^{gg'} - \zeta^{g'} r^{g'g} - \eta^g + \eta^{g'})\tag{53}$$

where F_{ext}^g is the external force on grain g . To solve the set of equations

$$\zeta_{ij}^g r_j^{gg'} - \zeta_{ij}^{g'} r_j^{g'g} = \eta_i^g - \eta_i^{g'}\tag{54}$$

we can rearrange the terms

$$(\zeta^g + \zeta^{g'}) (r^{gg'} - r^{g'g}) + (\zeta^g - \zeta^{g'}) (r^{gg'} + r^{g'g}) = \eta^g - \eta^{g'}\tag{55}$$

Thus, to make a simple start, assume that $\zeta^g \approx \zeta^{g'}$, which gives

$$\zeta^g = \frac{\partial \eta^g}{\partial r} R^{gg'}.\tag{56}$$

This equation has the structure

$$\vec{a} \cdot \vec{u} = b ,$$

whose general solution for \vec{u} is

$$\vec{u} = b \frac{\vec{a}}{a^2} + (\vec{a} \times \vec{c}) , \quad (57)$$

where \vec{c} is an arbitrary vector perpendicular to \vec{a} and \vec{u} . Thus, we expect

$$\zeta = \frac{\partial \eta}{\partial r} + \zeta^* , \quad (58)$$

where

$$\zeta_{ij}^{*g} r_i^{gg'} = \zeta_{ij}^{*g'} r_i^{g'g} . \quad (59)$$

Using the force balance conditions in eq. (50) gives

$$\frac{\partial \sigma_{ij}}{\partial x_j} = f_i^g + (\text{small corrections}) \quad (60)$$

whilst the second term can be further reduced by noting that the correct number of missing equations appears when (59) is summed over g' (i.e. there are still redundancies in (59) which complicate the analysis). From the condition

$$\sum_{g'} \zeta_{ij}^{*g'} r_j^{g'g} = 0 \quad (61)$$

one finds that σ must have the structure

$$\sigma_{ij}^g = \frac{1}{2} \sum (\phi_i^{g'} r_j^{gg'} + \phi_j^{g'} r_i^{gg'}) \quad (62)$$

where the elimination of the parameter ϕ leads to the missing equations [18]. Let us try a first approximation using the ansatz

$$\phi_i^{g'} = ar_i^{g'g} + br_i^{gg'} . \quad (63)$$

This gives the following averages

$$\langle \sigma_{ij}^g \rangle = \frac{1}{2} \sum_{g'} \left(2a \langle r_i^{gg'} r_j^{gg'} \rangle + b \langle r_i^{gg'} r_j^{g'g} + r_j^{gg'} r_i^{g'g} \rangle \right) , \quad (64)$$

leading to the simple form

$$\begin{vmatrix} \sigma_{xx} & C_{xx} + D_{xx} & E_{xx} + D_{xx} \\ \sigma_{xy} & C_{xy} + D_{xy} & E_{xy} + D_{xy} \\ \sigma_{yy} & C_{yy} + D_{yy} & E_{yy} + D_{yy} \end{vmatrix} = 0 \quad (65)$$

where

$$C_{ij} = \sum_{g'} R_i^{gg'} R_j^{gg'} \quad ; \quad D_{ij} = \sum_{g'} R_i^{gg'} S_j^{gg'} \quad ; \quad E_{ij} = \sum_{g'} S_i^{gg'} S_j^{gg'}$$

The crudest approximation one can use is $C_{yy} = C_{xx} = a^2$, $C_{xy} \approx 0$, $E_{ij} \approx 0$, $D_{xx} = -D_{yy} = \sin \theta$, $D_{yx} = \cos \theta$, where θ is the angle between \vec{R} and \vec{S} . This gives

$$\sigma_{xx} - \sigma_{yy} = 2 \tan \theta \sigma_{xy} \quad (66)$$

where θ varies in isotropic conditions within $-\pi < \theta < \pi$. From this expression one is prompted to define

$$\psi = \frac{\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} \quad (67)$$

which, for θ random, has a probability density

$$\mathcal{P}(\psi) = \frac{\psi}{\pi(1 + \psi^2)}. \quad (68)$$

Similarly, the probability density of ψ^{-1} is

$$\mathcal{P}(\psi^{-1}) = \frac{\psi^{-1}}{\pi(1 + \psi^{-2})}. \quad (69)$$

In sand piles, which are not isotropic, the form (66) has been successfully used to predict stresses throughout the pile, in particular reproducing the minimum under the apex [3].

A central difficulty of attempts to quantify granular physics is that these systems are very disordered. This means that, even when the existence of a statistical mechanical formalism is established, one must resort to approximations because a detailed statistical analysis is at present too difficult. Therefore, one should aim at this stage only for general laws. An example of such an approach is a recent exact formulation leading to an understanding of the onset of plasticity from a marginally rigid state of granular matter [20].

Second approach - coarse-graining a microscopic theory:

To derive the stress transmission equations on the granular level in two dimensions Ball and Blumenfeld [12] have followed a different, more rigorous, approach. Consider a two dimensional granular packing, part of which is shown in figure 4 together with its contact network, which has been defined above. The vectors \vec{r} connecting the contacts around grains circulate clockwise and therefore they form loops that circulate around the voids in the anti-clockwise direction. Ball and Blumenfeld defined for every such void loop a force \vec{f}_l located at the centre of the loop, which is defined as the mean position of the contacts that surround the loop. In two dimensions every contact, say between g and g' , sits on the boundary between two void loops, say l and l' . The grain g' exerts a force $\vec{f}_{gg'}$ on g and this force is parameterised in terms of the loop forces as follows

$$\vec{f}_{gg'} = \vec{f}_l - \vec{f}_{l'}. \quad (70)$$

The sign convention adopted in this expression is that if the vector $\vec{r}_{g'l}$ points towards grain g then \vec{f}_l is preceded by a positive sign and vice versa. The analysis of these forces, rather than the original contact forces, has several advantages:

1. By definition, the force exerted by grain g on g' is $\vec{f}_{g'g} = \vec{f}_{l'} - \vec{f}_l = -\vec{f}_{gg'}$. Therefore, the loop forces *satisfy Newton's third law of action and reaction*.
2. By writing the net force on grain g as a sum over its contact forces and representing the latter in terms of the loop forces, using (70), we observe that every loop force appears in the sum exactly twice, once with a positive and once with a negative sign. Thus, the net force on grain g vanishes identically, which means that the field of loop forces *automatically satisfies the force balance conditions on all the grains*. For consistency, the loop forces, just like the contact forces, must be uniquely determinate. If the system consists of $N (\gg 1)$ grains then there are $3N$ balance equations in total, two of force and one of torque moment per grain. From Euler's theorem on the topology of edges, vertices and cells in the plane, we have that there are on average six grain around a loop. Combining this with the fact that the mean coordination number is three it is immediate to deduce that there are $N/2$ voids in the packing. This gives that there are $N/2$ loop forces and therefore N unknowns to determine. This is to be contrasted with the $3N$ unknown components of the contact forces in the original force field. Since the loop forces already satisfy the force balance conditions then the loop forces can be only be determined through the torque balance equations. Of these there are exactly N , one per grain. Thus, there are exactly the same number of torque balance equations as there are unknowns and the loop forces are uniquely determinate.
3. The ratio of contact forces to loop forces is 3:1 and so the field of loop forces is three times more sparse than the original field of contact forces. Thus, a side effect of the parameterisation is that the field of loop forces is a coarse-grained version of the field of contact forces.

In terms of the loop forces and the vectors \vec{r} the force moment around grain g is

$$S_{ij}^g = \sum_l r_i^{lg} f_j^l \quad (71)$$

where the sum is over the loops that surround grain g . As mentioned above, the stress is the force moment normalised by a suitable area, and it is natural to use for this the grain area that consists of the areas of its quadrons, $A_g = \sum_q A_q$. Definition (71) is still discrete and we now need to pass to the continuum. This is done by a piecewise linear interpolation. We shall not describe this interpolation in detail, but an example of the interpolation of one component of the loop forces is shown in figure 8. Let us call the resulting continuous force field \vec{F} . In terms of \vec{F} the original forces are

$$\vec{f}_l = \vec{f}_g + \vec{R} \cdot \nabla F(\vec{x}_g) \quad (72)$$

Substituting this form into (71) the expression for the stress becomes

$$S_{ij}^g = \sum_l r_i^{lg} \left[\sum_k \vec{R}_k \partial_k F_j \right], \quad (73)$$

which can be written as

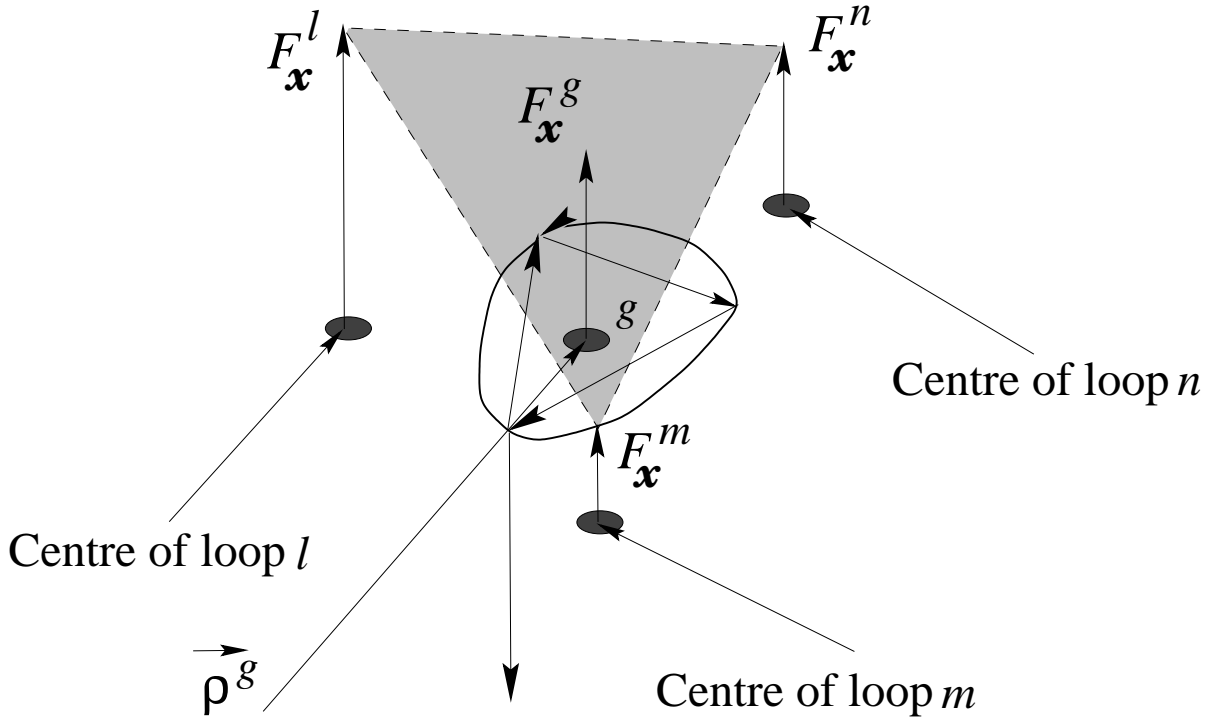


Figure 8: The piecewise linear continuation of the field of loop forces. Around a grain construct a polygonal surface (a triangular example is shown) whose corners are at $(\rho_x^l, \rho_y^l, f_i^l)$. The union of all such polygons is the continuous function F_i .

$$S_{ij}^g = \sum_l C_{ik} \partial_k F_j . \quad (74)$$

The geometric tensor \hat{C} is exactly the tensor defined in (39) which gave rise to the exact volume function in the entropic analysis. Thus, the tensor \hat{C} provides a natural characterisation of the geometry for the purpose of both the stress description and the entropy. Recalling that the stress tensor $\hat{\sigma}$ is the force moment properly normalised by area of a grain and integrating over a small region containing few grains, Ball and Blumenfeld were able to show that the imposition of torque balance gives *two* equations. One is

$$\langle \sigma_{ij} \rangle = \langle \sigma_{ji} \rangle ,$$

which corresponds to the global torque balance condition (50) for the mean stress tensor. The other condition is new

$$[\hat{P}\hat{\epsilon}^{-1}\hat{\sigma}]_{ij} = [\hat{P}\hat{\epsilon}^{-1}\hat{\sigma}]_{ji} , \quad (75)$$

where $\hat{P} = \frac{1}{2}(\hat{C} + \hat{C}^T)$, is the symmetric part of \hat{C} . Eq. (75) can be written explicitly as

$$p_{xx}\sigma_{yy} + p_{yy}\sigma_{xx} - 2p_{xy}\sigma_{xy} = 0 \quad (76)$$

and is a manifestation of the local torque balance condition. It gives a first-principles microscopic relation between the stress field and local geometric properties of the (generally disordered) microstructure, as characterised by the geometric tensor \hat{P} . Relation (76) is the missing constitutive equation for general disordered systems in two dimensions.

It is instructive to consider the statistical properties of \hat{P} . It has been shown [12] that the volume average of all its components, p_{ij} , vanishes identically *regardless* of the geometrical characteristics. This means that eq. (76) couples in fact between the stress field and *fluctuations* in the structural characteristics. The vanishing of the components p_{ij} under volume averaging presents a problem - it undermines conventional coarse-graining methods, which rely on finite-valued mean constitutive properties. This rather unique coarse-graining problem could, in principle, severely limit the applicability of the theory. Fortunately, this difficulty has been resolved by the development of a specialised procedure, based on concepts from frustration and antiferromagnetism [21]. An important result of that procedure is that the volume average of the coarse-grained p_{ij} became *finite*. This has made it possible to treat eq. (76) as macroscopic, opening the way to macroscopic calculations of the stresses.

The field equations (50) and (76) couple the stress components σ_{ij} . In a next development it was found that the equations can be resolved to yield explicit decoupled equations for each of these variables [22]. By making the assumption that for macroscopic scales the gradients of the fluctuations p_{ij} become small compared with the gradients of the stress field, general solutions have been obtained to the decoupled equations. The key to the analysis of the equations is a local linear transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{M}(\{p_{ij}\}) \begin{pmatrix} x \\ y \end{pmatrix}, \quad (77)$$

in terms of which the equations for the stress components become

$$\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) \sigma_{ij} = f_{ij}. \quad (78)$$

In this expression f_{ij} are source terms that depend only on the gradients of the external loading and on the local geometry. The differential operator on the left hand side is *hyperbolic*, confirming earlier suggestions [19] to this effect. Eq. (78) is quite elegant in that all the stress components follow the same equation but with different source terms f_{ij} . The general solutions give rise to force chains that propagate through the isostatic granular medium [22], in good agreement with experimental observations [23]. The form (78) also made it straightforward to derive the Green function, resolving a long debate in the literature. It has been further shown recently by one of us that introducing corrections due to the gradients of the constitutive parameters p_{ij} does not affect the force chains, leading to the conclusion that, at least in two dimensions, the force chains are a generic solution of the general stress field equations and that they follow the characteristic curves $\eta = u + v$ and $\zeta = u - v$.

The above analysis has been limited to planar systems. A promising extension of the theory to three-dimensions is under construction by one of us [25] at present, using insight from the two-dimensional case. However, it is unclear at this stage whether the three-dimensional theory would also give rise to force chain solutions.

The experimental status is also not entirely clear. Detailed measurements by Brujic [26] in emulsions, where the colloidal particles exert force on one another, show no sign of force chains within the bulk. This may indicate either that force chains are not as prominent in three dimensions as they are in two or that they are negligible for very soft particles. Alternatively, this may point to inherent differences between systems of particles with very little, or no, friction, which emulsions are, and systems of particles with friction between them. Intriguingly, it has been suggested in [12] on the grounds of theoretical stress analysis that even in two dimensions there may exist an inherent difference between systems of smooth and rough grains, but how could such a difference manifest in the elimination of force chains, if it does, is unclear at present.

§5 Force distribution

There is a distinct difference between the problem of force statistics and the distribution in configurations space. The packing of grains on the reversible branch is governed by the function $e^{-W/X}\Theta$. The forces have little effect on this distribution, but they are fully dependent on the configurations and the external loading. For a given structural configuration, the problem is the following. Given external, i.e. surface, loading and internal, i.e. body, forces what is the solution of the balance conditions (28) and (29)?

We first recall that for infinitely rigid grains there can only be a statically determinate solution if the mean number of contacts between grains takes a particular value, z_c . In d dimensions this value is $d+1$ for rough grains, $d(d+1)$ for smooth arbitrary grains, and $2d$ for smooth spheres. These values are obtained upon requiring that in mechanical equilibrium the number of force and torque balance conditions should be equal to the number of unknown force components. Here we discuss only the case of rough grains and we shall assume that this condition is always obeyed. Several objections can be (and have been) raised to this assumption: one is that real systems are never fully rigid and another is that the number of contacts in real granular packings is usually higher than z_c , undermining the determinacy condition. The first objection has been shown recently to be misguided: the infinite rigidity is not a necessary condition and packings of compliant grains that satisfy the coordination number condition are also isostatic [2]. The second objection presents an interesting challenge. It is true that in most real granular packings $z > z_c$ even if the grains are very hard. In fact, a recent experiment has identified that upon consolidation of a granular pile the coordination number can only approach z_c as the consolidating grains approach a particular density and therefore that there exists a marginally rigid state where $z = z_c$ [27]. The question, however, is how large above z_c does z need to be for the isostatic behaviour (and for the isostaticity theory discussed above) to become irrelevant. Clearly, this cannot take place abruptly - adding one contact in an otherwise isostatic macroscopic system would not suddenly change the stress field across the entire system from a solution of the isostaticity equations to a solution of elasticity theory. Rather, the change must be gradual. A preliminary discussion of the manner that the change might occur has been presented recently [22] with the conclusion that the isostatic theory only deteriorates gradually as the number of extra contacts increases. This new understanding, while exciting, is out-

side the scope of this presentation. To simplify the following discussion, and without loss of much generality, we shall assume that all grains have exactly $d + 1$ contacts. The generalisation to more general distributions is straightforward although more tedious.

In a macroscopic system the forces will have a distribution that, for a homogeneous powder under external pressure, may be roughly like a hydrostatic pressure p : $\vec{f} \approx -p\vec{n}$, where \vec{n} is a coarse-grained normal, or $f \approx p$. The distribution of f has been observed in several experiments [23] to be exponential and several theoretical models have been proposed to explain it. The derivation of an exact equation is quite difficult if one attempts to take into account random shaped grains or even a random distribution of grain contacts. Therefore, the present discussion will start generally and then we shall simplify the internal structure of the equation to allow analytic solutions.

Let us use the tensor $\hat{\mathcal{F}}^g$ and in particular base the analysis on $\text{Tr}[\hat{\mathcal{F}}^g]$ which will relate to the average

$$\langle f \rangle = \frac{1}{2} \sqrt{\text{Tr}[\hat{\mathcal{F}}]} . \quad (79)$$

The tensor $\hat{\mathcal{F}}^g$ is related to the touching grains g_1, g_2, g_3, g_4 via Newton's equations and therefore its probability density satisfies

$$\mathcal{P}(F^g) = \int \delta \left(F^g - \sqrt{\text{Tr}[\hat{\mathcal{F}}^g]} \right) \Pi_i \mathcal{P}(F^{g_i}) dF^{g_i} \Phi([\mathcal{F}]) \quad (80)$$

where Φ is a set of δ -functions that fix the relationships of the variables F^{g_i} . We can then use the crude approximation that F^g is linearly related to F^{g_i} with coefficients λ^{g_i}

$$\mathcal{P}(F) = \int_0^1 \dots \int_0^1 \delta \left(F - \sum_{g'=1}^d \lambda^{g'} F^{g'} \right) \Phi([\lambda^{g'}]) \Pi_{g'} [\mathcal{P}(F^{g'}) dF^{g'}] . \quad (81)$$

The Fourier transform of this expression gives

$$\mathcal{P}(K) = \int \dots \int \Phi([\lambda^{g'}]) \Pi_{g'} [\mathcal{P}(\lambda^{g'} K) d\lambda^{g'}] \quad (82)$$

The limits 0, 1 originates in the definition of F above. The simplest approximation is to put $\Phi = \text{constant}$, which gives

$$\frac{dQ}{dK} = \frac{Q^d}{K^d} \quad (83)$$

where $\mathcal{P}(K) = dQ/dK$. Solving for Q and differentiating we obtain

$$\mathcal{P}(K) = \frac{1}{(1 + cK^{d-1})^{d/(d-1)}} . \quad (84)$$

In two dimensions the inverse transform gives

$$\mathcal{P}(F) = \frac{\text{Constant}}{p^2} F e^{-F/p} \quad (85)$$

whilst in three dimension we can use fractional derivatives to express the Fourier transform

$$\mathcal{P}(F) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iKF} dK}{(1 + p^2 K^2)^{3/2}} \quad (86)$$

as a sum over Bessel functions. For small K (large F) we get that

$$\mathcal{P}(F) \approx \frac{1}{(2/3)!} \left(\frac{F}{p}\right)^{2/3} e^{-F/p} \quad (87)$$

Different approximations for the function Φ may alter the power 2/3 but the exponential structure is robust. Different experiments by Liu [24], Brujic [26] and analyses [28], all agree with the exponential form. One can always question the error involved in the approximation of the function Φ as a constant, but whether it is worth the effort of obtaining a more accurate expression given the randomness of grain shapes or even the topology of the contact network is debatable at this stage. One thing seems clear from the experimental evidence - $\mathcal{P}(F)$ rises from a very small value, possibly zero, peaks and then decreases exponentially towards large forces, and this feature is captured by this simple model.

The canonical distribution of stresses mentioned above offers an alternative to the direct self-consistent approach for it predicts

$$e^{-\frac{1}{v_{grain}} \sum f_i r_i \frac{\partial S}{\partial P}} \quad (88)$$

where $f_i r_i$ is the diagonal element of the force moment and we are treating only the pressure part of the stress for simplicity. If \vec{r} is roughly parallel to \vec{f} the distribution (88) behaves as $e^{-c|\vec{f}|}$. This brief derivation needs to be refined by the fact that the neighbouring \vec{f} 's are related and Newton's laws. This is a new approach which is currently being explored.

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