Exact multi-twist solutions for Heisenberg spins on an elastically deformable cylinder

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We study the magnetoelastic behavior of a system of classical Heisenberg spins on an elastic cylinder. By applying a nonlinear transformation we solve the self-dual equations associated with the magnetic part. We obtain a novel hierarchical family of exact solutions that describe multi-twist spin configurations on the surface of the elastic cylinder.

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The role of nonlinear excitations in low-dimensional, artificially structured materials is becoming increasingly important due to their observable effects on the physical properties of realizable condensed matter systems. The underlying physics becomes much richer if the interplay of topology and (curved) geometry is also taken into account. Here we address this issue and find a new *hierarchy* of domain-wall-type nonlinear excitations for the classical Heisenberg spins on the surface of an elastic cylinder. These excitations are quite different from the known magnetic solitons or skyrmion-like solutions.

The motivation for seeking new exact solutions comes from a seven year old prediction that, for classical Heisenberg spins on an infinite cylinder, a periodic topological spin soliton and/or an anisotropic spin-spin coupling should produce a deformation of the cylinder in the region of the soliton due to violation of the self-duality equations [1], which are satisfied, however, for the isotropic single soliton case. The solutions of the self-dual equations are the absolute minimum of the energy in each homotopy class associated with a spin distribution, where a homotopy class is characterized by its winding number. All solutions of these equations satisfy the Euler-Lagrange equation, but not vice versa. The violation of self-duality is closely related to the concept of geometrical frustration: A misfit between the width of the soliton (characteristic magnetic length) and the radius of the cylinder (characteristic length of the system).

It has been anticipated that elastic cylinders would deform due to magnetic interactions [1] but explicit solutions could not be obtained for any deformation of the cylinder due to the nonlinear nature of the resulting equations. Assuming a *rigid* cylinder and cylindrical symmetry it was found that the single spin twist solution $\theta = 2\arctan\exp(x/\rho_0)$ is the soliton solution of the sine-Gordon equation and the energy associated with this solution $H = 8\pi J|Q|$ is independent of the radius ρ_0 of the rigid cylinder. Here Q is the winding number (or the Pontryagin index) of the spin solution ($Q = 1$ for the single twist soliton). Since the sine-Gordon equation supports a periodic solution it was suggested that a periodic pinch would occur on the cylinder, were it deformable.

Here we dispose of both the assumptions of cylindrical symmetry and rigidity, $\rho = \rho_0$, and allow elastic deformations of the cylinder in the solution of the equations. We employ a nonlinear transformation that gives an *implicit* separation of the magnetic part of the Hamiltonian from the elastic in a deformed geometry that corresponds to the exact elastic solution. Using the transformation we then obtain a family of exact solutions, each corresponding to a set of twists in the spin field. The solutions are given in a metric that depends explicitly on the deformation of the cylinder and is valid for any continuous deformation. This formalism holds for general underlying geometries and is not particular to the cylindrical support on which we focus here. Physically relevant systems include magnetically coated deformable (whether metallic, non-metallic or organic) cylindrical thin films on which our model and the presence of the predicted spin domain walls can be tested experimentally.

The Hamiltonian is given by $H = H_{magn} + H_{el} + H_{m-el}$ where the magnetic part (the nonlinear σ model) in the cylindrical coordinates is

$$H_{mag} = J \int\int dx d\tau \rho \sqrt{1+(\partial_x \rho)^2} \times \left[\frac{(\partial_x\theta)^2}{1+(\partial_x\rho)^2} + \frac{\sin^2\theta}{1+(\partial_x\rho)^2}(\partial_x\phi)^2 + \frac{(\partial_\tau\theta)^2}{\rho^2} + \frac{\sin^2\theta}{\rho^2}(\partial_\tau\phi)^2\right] \quad (1)$$

and the elastic part is

$$H_{el} = \int\int dx d\tau \rho \left[\frac{\chi}{4}(\frac{d\rho}{dx})^2 + \frac{\chi'}{\rho_o^2}(\rho-\rho_o)^2 + \frac{\chi''}{4}\frac{(\partial_\tau\rho)^2}{\rho^2}\right] \quad (2)$$

Here χ, χ' and χ'' are elastic constants of the cylinder for deformation along the axial (x), radial (ρ) and azimuthal (τ) directions, respectively. In eq. (1) J denotes coupling energy between neighboring spins, and the magnetization unit vector $\hat{n} = (\cos\theta,\ \sin\theta\ \cos\phi,\ \sin\theta\ \sin\phi)$. Bending and torsion contributions to the elastic energy have been neglected in Eq. (2). The magnetoelastic coupling energy is given in general by $H_{m-el} = \Sigma_{iklm}\Gamma_{iklm}\sigma_{ik}n_l n_m$ [2, 3], where Γ_{iklm} is a symmetric (with respect to interchanging the pairs i, k with l, m) tensor of rank four and σ_{ik} is the stress tensor [4, 5]. For 1D spin chains

H_{m-el} couples nonuniform chain stretching to the magnetic part, leading to J variation along the chain. In this case there exists an important result [6]: to linear order, H_{m-el} can be absorbed into H_{magn}, merely renormalizing J. In what follows we assume that the same applies in 2D for stretching in both directions [7].

The above discussion holds for any arbitrary smooth surface S with infinitesimal surface element $d\vec{S} = \sqrt{|g|}\ \ d\Omega$, where $\sqrt{|g|}$ is the determinant of the metric tensor $g^{\mu\nu}$ of the support. Furthermore, to all such surfaces one can apply an inequality due to Belavin and Polyakov [8] and Bogomol'nyi [9]),

$$The\ inequality\ ,\ specific\ to\ our\ geometry \tag{3}$$

For a deformed cylinder [4, 5] $\sqrt{|g|} = \sqrt{\rho^2 + (\partial_\tau \rho)^2}\sqrt{1+(\partial_x \rho)^2}$. In Eqs. (1) and (2) we have assumed that χ'' is sufficiently large such that $\partial_\tau \rho \ll \rho$ and $\chi''(\partial_\tau \rho)^2$ is finite [of the order of the first two terms in Eq. (2)].

Let us now define a new coordinate ζ and a new variable ψ through the nonlinear transformation

$$d\zeta = \frac{\sqrt{1+(\partial_x \rho)^2}}{\rho} dx \quad ; \quad d\psi = \frac{d\theta}{\sin\theta} \tag{4}$$

The fields ψ, ϕ and ρ now become functions of the coordinates ζ and τ and the magnetic and elastic parts of the Hamiltonian in the transformed coordinate system can be written as:

$$H_{magn} = J \int\int \frac{|\nabla\psi|^2 + |\nabla\phi|^2}{\cosh^2\psi} d\zeta d\tau \tag{5}$$

$$H_{el} = \int\int d\zeta d\tau\ \rho\sqrt{\rho^2 - (\partial_\zeta \rho)^2} \times \left[\frac{\chi}{4}\frac{(\partial_\zeta\rho)^2}{\rho^2 - (\partial_\zeta\rho)^2} + \frac{\chi'}{\rho_o^2}(\rho-\rho_o)^2 + \frac{\chi''}{4}\frac{(\partial_\tau\rho)^2}{\rho^2}\right] \tag{6}$$

where the gradient ∇ is in the ζ-τ plane and we have used (3a) to replace

$$\partial_x \rho = \frac{\partial_\zeta \rho}{\rho}\sqrt{1+(\partial_x\rho)^2} = \frac{\partial_\zeta\rho}{\sqrt{\rho^2 - (\partial_\zeta\rho)^2}} \tag{7}$$

Note that the magnetic part, Eq. (5), describes a free particle Hamiltonian with a field-dependent mass. Note also that Eq. (5) depends on ρ implicitly through ζ.

The Euler-Lagrange equations for the magnetic variables can now be derived and studied. Similarly, the Euler-Lagrange equation for the elastic part can be solved as a function of x and τ (either analytically or numerically) given ψ and ϕ. This, however, is not our aim here. Rather, we are interested in finding exact magnetic solutions for arbitrarily inhomogeneous configuration of spins on the boundaries of the cylinder.

Traditional treatments of the problem assume uniform such boundary conditions, giving rise to a well defined integer winding number Q. This number is the value of the integral on the right hand side of Eq. (3) over the entire surface normalized by $8\pi J$. For nonuniform boundary conditions the integral generically yields a 'fractional' winding number, q. In traditional treatments the class of solutions with a particular value of Q are said to be in the same homotopy class and, of those, the lowest energy solution satisfies (3) as an equality. In terms of the angles of the spin vector this gives the self-dual equations [8][9]:

$$\begin{aligned} \sin\theta\partial_\tau\phi &= \pm\frac{\rho}{\sqrt{1+(\partial_x\rho)^2}}\partial_x\theta \\ \frac{\sin\theta}{\sqrt{1+(\partial_x\rho)^2}}\partial_x\phi &= \mp\frac{\partial_\tau\theta}{\rho} \end{aligned} \tag{8}$$

The same discussion applies to the case of nonuniform boundary conditions, with equality giving the lowest energy state in the q-family of solutions.

In the new variables and coordinates Eqs. (8) conveniently reduce to Cauchy-Riemann relations between the real functions, ψ and ϕ,

$$\psi_\zeta = \pm\phi_\tau \quad ; \quad \psi_\tau = \mp\phi_\zeta \tag{9}$$

The general solutions of Eqs. (9) are the harmonic functions plus a logarithmic singularity in the presence of sources (charges). On the cylinder these solutions must be periodic in τ and should tend to predetermined boundary conditions at both ends of the cylinder. It follows that

$$\begin{aligned} \cos\theta_N &= \frac{n}{2}\ln(\zeta^2+\tau^2) + \tanh \\ &\left[\sum_{k=1}^{N}\left[\alpha_k\cos(k\tau)+\beta_k\sin(k\tau)\right]\left[a_k\cosh(k\zeta)+b_k\sinh(k\zeta)\right]\right] \\ \phi_N &= \frac{n}{2}\arctan(\tau/\zeta) + \\ &\sum_{k=1}^{N}\left[\alpha_k\sin(k\tau)-\beta_k\cos(k\tau)\right]\left[a_k\sinh(k\zeta)+b_k\cosh(k\zeta)\right] \end{aligned} \tag{10}$$

The term outside the sum in both expressions corresponds to the well-known instanton and is the only one that survives under uniform boundary conditions. Only nonuniform boundary conditions bring the higher harmonics to life. The coefficient n (equivalent to the n-instanton) is, in our terms, the 'charge' of the instanton. Expressing the above solutions in terms of θ and phi, there appear local twists in $\cos\theta$-field wherever ψ vanishes, as we proceed to illustrate.

The instanton sloution, with $n=1$, is shown in figure 1, where we plot the value of $\cos\theta$ over the entire cylinder. In terms of ψ, this solution diverges at $\pm\infty$ (corresponding to $\cos\theta \to \pm 1$ or $\theta \to 0, \pi$). The solution also diverges logarithmiocally in the vicinity of the charge, as expected. The divergence at the source is due to a positive (negative) charge, corresponding to $\cos\theta = \pm 1$ and physically originates in a group of spins that are fixed at a particular orientation. The bigger this group, the larger the local charge.

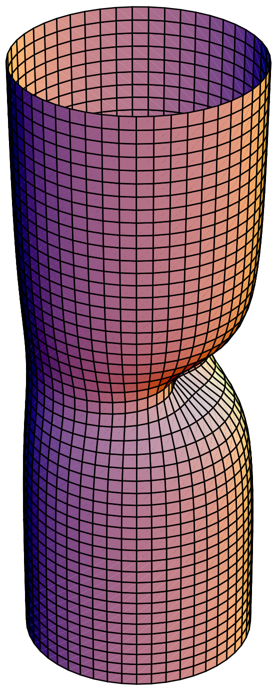

FIG. 1. An instanton solution for $\cos\theta$ with $n = 1$, being the instanton charge in eq. (10), and with all other terms vanishing. Shown is the surface formed by the tip of the spin on top of a rigid cylinder of radius R=2 in the $\zeta - \tau$ coordinates. Note that in the physical coordinates, $x - \tau$, the cylinder is also elastically distorted. The divergence of the function ψ both toward the ends of the cylinder and at $(\zeta, \tau) = (0, 0)$, corresponds to $\cos\theta = 1$. Notice the uniform boundary conditions.

Turning to the twist solutions, in figures 2-4 we plot solutions consisting of the first, second and third harmonics alone. These show the twists between magnetic regions with exactly opposite orientations. The twists are located along the lines of nodes of the function ψ. The maximal number of domain walls, counting around the cylinder at any possible ζ, is twice the order of the highest mode in eqs. (10), N. Therefore, it is the boundary conditions that mostly determine the structure. The more modes ψ consists of, the richer the texture of the twists. We emphasize that the coordinate system underlying these solutions is τ-ζ. To convert the plots to the τ-x coordinates one needs to solve explicitly the Euler-Lagrange equations for $\rho(x)$. This class of exact solutions is new in the cylindrical geometry. To the best of our knowledge, even for the rigid cylinder ($\partial_x \rho = 0$, $\zeta = x/\rho_0$) this is the first time that these nonlinear textures have been derived.

The magnetoelastic term in the Hamiltonian will couple the magnetic twists to the stress field and will give rise to deformations of the cylinder. This can be quantified by writing down the Euler-Lagrange equations explicitly (not presented here) in the original coordinates [5]). Although these equations are too cumbersome to handle analytically, a simple substitution would convince the reader that $\rho = \rho_0$ *is not a general solution*, except in the trivial case when the magnetic variables θ and ϕ are constant.

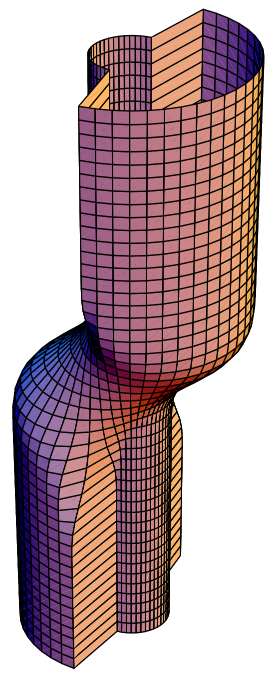

FIG. 2. The spin field of a lowest mode solution, in the $\zeta - \tau$ coordinates. This solution corresponds to $\beta_1 b_1 \neq 0$, while all the other terms vanish. The two nodes of this mode generate a double-twist around the cylinder. Note two marked difference between the instanton and the twist solution: First, the boundary conditions at the edges of the cylinder are no longer uniform. Second, the twist represents an exponentially sharp walls between domains of exactly opposite spin polarity.

Equations identical in form to (8) are also obtained in the context of (i) low-energy dynamics of the classical, isotropic, continuous antiferromagnetic Heisenberg chains [10], and (ii) the stationary Landau-Lifschitz equation in the context of two-dimensional ferromagnetic Heisenberg model. In these contexts, as here, these solutions are distinct from the n-instanton (skyrmion) [8] and meron [11] classes. The connection between the dynamic solutions for antiferromagnetic chains, stationary solutions for ferromagnetic planes, and the above solutions for deformable cylinders is useful for understanding the behavior of all these systems from analysis of any one of them.

In conclusion, through a nonlinear transformation of the coordinates and the variables we have rewritten the Hamiltonian of the nonlinear σ model on an elastic cylinder in terms of spin-related fields, ψ, ϕ, and an elastic-related one ρ separately. The resulting equations resemble the self-dual Belavin-Polyakov equations in all but the boundary conditions, which need not be uniform. With the help of the transformations, these equations become exactly solvable. We have constructed a new explicit class of exact spin twist solutions for states with fractional winding numbers. These solutions and associated magnetoelastic effects (deformation of the cylinder) are important in their own right as well as in the theory of nonlinearity and dynamics. We emphasize that although we illustrated the results on a deformable cylinder, the formalism is valid for a general geometry (with the metric tensor $g^{\mu\nu}$), e.g., on a sphere, a torus, etc.

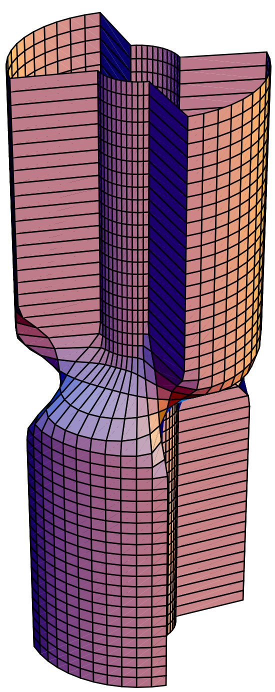

FIG. 3. The spin field of a second mode solution, in the $\zeta - \tau$ coordinates, corresponding to $\beta_2 b_2 \neq 0$, while all the other terms vanish. This solution has four nodes, giving rise to four domain walls around the cylinder.

Finally, we suggest that it should be possible to experimentally observe these magnetic solutions, the winding number q and the concomitant magnetoelastic deformations in real systems such as magnetically coated cylindrical thin films, possibly on recently observed tubular fluid membranes [12], and even on nanotubes. Specifically, ultrasonic techniques (such as change in normal modes of the cylinder subsequent to domain wall formation, attenuation and phase shift) can probe the deformation of the cylinder, while magnetic force microscopy (MFM) can be employed to observe the magnetic texture. Relevant systems include: cylindrically wrapped thin films of magnetic materials, e.g., layered 2D Heisenberg magnets such as $(C_nH_{2n+1}NH_3)_2MX_4$ and $[NH_3(CH_2)NH_3]MX_4$ for n$\leq$16, where M = Cr, Mn, Fe, Cu, Cd and X = Cl, Br [13]. Other examples include K_2CuF_4, Ca_2MnO_4, Rb_2FeF_4, etc. [13] and magnetic Langmuir-Blodgett films of manganese stearate $Mn(C_{18}H_{35}O_2)_2$ [14].

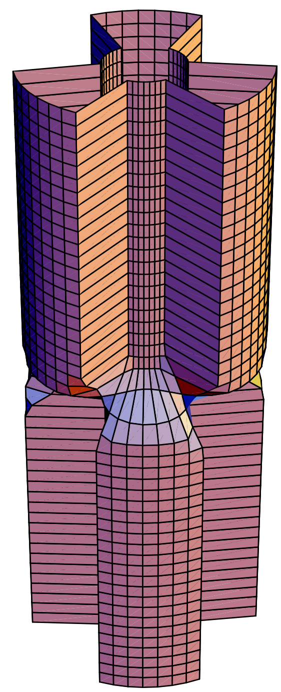

FIG. 4. As in firgures 2 and 3, the spin field of a third mode solution, in the $\zeta - \tau$ coordinates. The solution corresponds to $\beta_3 b_3 \neq 0$, while the other terms vanish, and gives rise to six domain walls around the cylinder.

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[3] W. F. Brown, Jr., *Magnetoelastic Interactions* (Springer-Verlag, New York, 1966); E. Callen, J. Appl. Phys. **39**, 519 (1968). Alternatively, $H_{m-el} = -2\sum_{i>j} \varepsilon_{\alpha\beta} x_\alpha \partial J(r_{ij})/\partial x_\beta \ (n_i.n_j)$, where $\varepsilon_{\alpha\beta}$ are components of the strain tensor.

[4] See, e.g., E. A. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Dover, New York, 1944).

[5] In cylindrical coordinates $H_{m-el} = \sigma_{\rho\rho}[\sin\theta\cos(\phi-\tau)]^2 + 2\,\sigma_{\rho x}\sqrt{1+\rho_x^2}\sin\theta\cos(\phi-\tau)[\cos\theta+\rho_x\sin\theta\cos(\phi-\tau)]$
$+\,2\,\sigma_{\rho\tau}\sqrt{\rho^2+\rho_\tau^2}\sin^2\theta\cos(\phi-\tau)[\rho_\tau\cos(\phi-\tau)+\rho\, sin(\phi-\tau)] + \sigma_{xx}(\rho^2+\rho_\tau^2)[\cos\theta+\rho_x\sin\theta\cos(\phi-\tau)]^2$
$+\,2\,\sigma_{\tau x}\sqrt{\rho^2+\rho_\tau^2}\sqrt{1+\rho_x^2}\sin\theta[\rho_\tau\cos(\phi-\tau)+\rho\sin(\phi-\tau)][\cos\theta+\rho_x\sin\theta\cos(\phi-\tau)]$
$+\,\sigma_{\tau\tau}(\rho^2+\rho_\tau^2)\sin^2\theta[\rho_\tau\cos(\phi-\tau)+\rho\sin(\phi-\tau)]^2$.

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