Stress chain solutions in two-dimensional isostatic granular systems: fabric-dependent paths, leakage and branching

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The stress field equations for two-dimensional disordered isostatic granular materials are reformulated, giving new results beyond the commonly accepted force chains. Localized loads give rise to exactly determinable cones of influence, bounded by stress chains. Disorder couples same-source chains, attenuates stresses along chains, causes stress leakage from chains into the cone, and gives rise to branching. Chains from spatially separate sources do not interact. The formulation is convenient for computation and several numerical solutions are presented.

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Natural, technological and theoretical challenges made granular materials a focus of much attention[1]. Of particular interest is their mechanical behaviour, following experimental [2–5] and numerical [6] observations of nonuniform stress transmission. Conventional elasticity theory can only explain this phenomenon by imposing anisotropic elastic constants rather than directly from the stress equations. Alternatively, it has been proposed that this is the result of isostatic, or statically determinate, regions that grow as granular media approach marginally rigid states [7–9], e.g. before and after failure or near the rigidity percolation threshold [10]. This picture is supported by numerical results [11]. A continuous theory of stresses of isostatic states is still in its infancy but equations have been proposed and derived in several approximations [8, 9, 12–15]. In 2D, the stress balance equations are given by the two force equations

$$\partial_x \sigma_{xi} + \partial_y \sigma_{yi} = g_i \quad ; \quad i = x, y \tag{1}$$

and a global torque moment, $\sigma_{xy} = \sigma_{yx}$. $\vec{g} = (g_x, g_y)$ is an external force field, including body forces. In isostaticity theory, the equations are closed by a linear stress-structure relation

$$p_{xx}\sigma_{yy} + p_{yy}\sigma_{xx} = 2p_{xy}\sigma_{xy} \ . \tag{2}$$

The form (2) had been suggested originally on empirical basis [12, 14, 15] and later derived from contact mechanics [16]. The latter provided a geometric interpretation of P, which is symmetric and whose components are p_{ij}, establishing that: (i) the mean of $\mathrm{Tr}P$ over all grains vanishes; (ii) $\det P < 0$ above some scale; (iii) the stress equations are hyperbolic, as previously hypothesized [12]. The advantage of this approach is that it enables derivation of force chains direction without imposing a-priori anisotropic properties or principal stress axes. The correspondence between the continuous stress chains solutions and force chains observed experimentally has been made in reference [8]. Stress chain solutions for coarse-grained [17] constant fabric parameters have been derived in [8, 9] by decoupling eqs. (1)-(2) into an integro-differential form. Attempted perturbation analysis about constant P had limited success [9].

Here we derive a range of new phenomena for disordered systems directly from the field equations. This is made possible by re-formulation of eqs. (1)-(2) as a strictly hyperbolic first order system. We derive analytical results and demonstrate the convenience of the new formulation for numerical calculations. Specifically, we find that: chains from different sources pass through one another unaffected, isostatic stresses exhibit coning, coupling between same-source chains, intra-cone leakage, and branching. In particular, this is the first explicit derivation of force chain leakage and branching directly from the governing isostaticity equations.

Consider first the general case $p_{xx} \neq 0$. The analysis for $p_{xx} = 0$ is simpler, leading to similar qualitative results. Substituting σ_{yy} from (2) into (1) yields

$$\partial_x \vec{u} + \partial_y (A\vec{u}) = \vec{g}, \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ -q_{yy} & 2q_{xy} \end{pmatrix} \ , \tag{3}$$

where $q_{iy} = p_{iy}/p_{xx}$ $(i = x, y)$ and $\vec{u} = (\sigma_{xx}, \sigma_{xy})$. The eigenvalues of A are

$$\lambda_{1,2} = q_{xy} \pm \sqrt{q_{xy}^2 - q_{yy}}. \tag{4}$$

On the granular level, the trace of P fluctuates around zero [16] and therefore q_{yy} is predominantly negative. Since $\det(P) < 0$ then $q_{xy}^2 - q_{yy} = -\det(P)/p_{xx}^2 > 0$, and the eigenvalues λ_i are distinct and real. It follows that the system (3) is indeed strictly hyperbolic.

Consider first constant fabric tensors and define the characteristic variables

$$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 \sigma_{xx} - \sigma_{xy} \\ -\lambda_1 \sigma_{xx} + \sigma_{xy} \end{pmatrix} \ . \tag{5}$$

Eq. (3) decouples into

$$\partial_x w_i + \lambda_i \partial_y w_i = h_i, \ i = 1, 2 \ , \tag{6}$$

where λ_i are given by (4) and

$$\vec{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 g_x - g_y \\ -\lambda_1 g_x + g_y \end{pmatrix} . \quad (7)$$

We now parameterize the characteristic paths by s via

$$dx/ds = 1 \ , \quad dy/ds = \lambda_i \ , \quad i = 1, 2 \ . \quad (8)$$

When P is constant the characteristics are straight lines. Since $q_{yy} < 0$ the characteristics have gradients of opposite signs. Using definition (8), eqs. (6) can be written as ordinary differential equations

$$dw_i/ds = h_i \ , \quad i = 1, 2 \ . \quad (9)$$

along the characteristic paths.
Consider the semi-infinite plane $x \geq 0, -\infty \leq y \leq \infty$. At any point $(\widetilde{x}, \widetilde{y})$, the characteristic variables $\vec{w}$ can be computed by integrating eqs. (9), each along its unique characteristic path,

$$w_i\left(\widetilde{x}, \widetilde{y}\right) = w_i\left(0, \widetilde{y} - \lambda_i \widetilde{x}\right) + \int_0^{\widetilde{x}} h_i(s) ds \ , \quad i = 1, 2 \ . \quad (10)$$

Substituting (10) into (5) yields σ_{xx} and σ_{xy}. σ_{yy} is then determined from eq. (2).

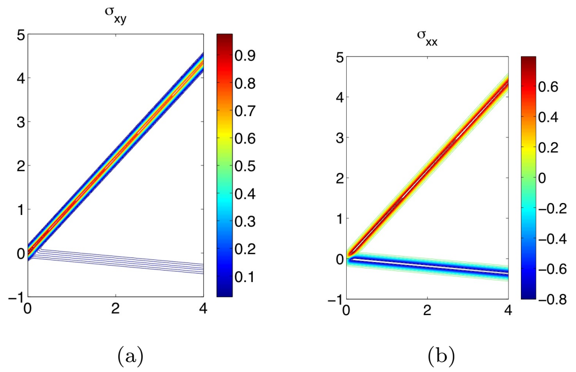


FIG. 1: The full numerical solution for the stress field in a semi-infinite plane, whose fabric tensor is given by (11), under a narrow Gaussian-shaped source of σ_{xy} applied to the boundary $x = 0$. We show contour plots of σ_{xy} (a) and σ_{xx} (b). Contour colors give the stress magnitude. Note that σ_{xx} is non-zero even with no boundary load in this component.

For simplicity, we set $g_x = g_y = 0$ and consider in the following only systems loaded through the external boundaries by narrowly localized sources. The extension to general source terms and finite external fields $\vec{g}$ is straightforward. Solution (10) shows that applying a localized stress at the boundary gives rise to stresses that propagate into the system only along the two characteristic paths. These are stress chains. The paths straddle a 'cone of influence' [9]. For constant fabric tensors, the characteristic variables, and hence the stresses, do not attenuate along these paths. This reproduces the stress chains found in [8, 12]. Also evident from solution (10) is that: (i) the stresses vanish at points not connected by a characteristic path to a non-zero boundary source; (ii) stresses emanating from different boundary locations pass through one another unchanged.
No experimental or numerical data exist currently for the tensor P, which characterizes local rotational disorder [16]. Therefore, we illustrate our results on synthetic cases. In fig. 1 we show the full numerical solution for a system whose fabric tensor is

$$P = \begin{pmatrix} 1 & , 1/2 \\ 1/2, & -3 \end{pmatrix} \ , \quad (11)$$

when a narrow Gaussian-shaped stress σ_{xy} is applied to the boundary at $x = 0$. All the numerical stress solutions in this paper were obtained by solving eqs. (6) for the w_i with a standard upwind finite difference method [18]. Fig. 1a shows the solution for σ_{xy}. Fig. 1b shows that even without loading σ_{xx} along the boundary, σ_{xx} is non-zero along the characteristic paths. This is because the value of w_i must remain constant along the path.
We next apply this approach to more realistic systems of variable fabric tensors. Repeating the steps leading to (6), the characteristic variables w_i, defined again by eq. (5), satisfy the equations

$$\partial_x w_i + \lambda_i \partial_y w_i = h_i - \partial_y \lambda_i w_i + (B\vec{w})_i \ , \quad (12)$$

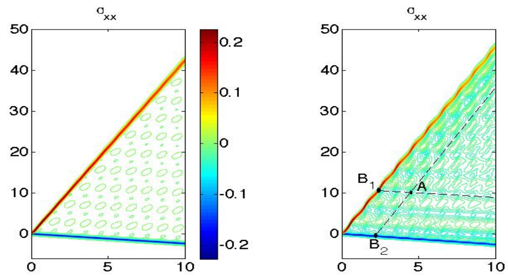
where the matrix B depends on P and its gradients,

$$B = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \partial_x \lambda_1 + \lambda_1 \partial_y \lambda_1 & \partial_x \lambda_2 + \lambda_2 \partial_y \lambda_2 \\ -\partial_x \lambda_1 - \lambda_1 \partial_y \lambda_1 & -\partial_x \lambda_2 - \lambda_2 \partial_y \lambda_2 \end{pmatrix} . \quad (13)$$

Form (12) allows us to make several significant observations and it is convenient for numerical analysis. Following are four key observation, which we support by full numerical solutions, using for illustration

$$P = \begin{pmatrix} 1, 2 \\ 2, -1 \end{pmatrix} + \begin{pmatrix} \varepsilon_{11}, \varepsilon_{12} \\ \varepsilon_{12}, \varepsilon_{22} \end{pmatrix} \cos(kx) \ . \quad (14)$$

1. The characteristic variables are now governed by $\partial_x w_i + \lambda_i \partial_y w_i = dw_i/ds$. Stresses propagate along characteristic paths still defined by (8), but with λ_i functions of position. Consequently, the stress paths are no longer straight and (8) must be integrated to compute them. This is illustrated in fig. 2a for the full solution for $\varepsilon_{ij} = 0.1$, where the characteristic paths waiver slightly about straight lines. We will continue to call the region that the paths straddle cone of influence.
2. From (12) we see that stresses may attenuate along the paths due to the effects of $\partial_y \lambda_i$ and the diagonal terms B_{ii}. This phenomenon is observed in the full solution for σ_{xy} for $\varepsilon_{ij} = 0.5$ and $k = 2\pi$, when a narrow Gaussian-shaped σ_{xx} is applied to the boundary (fig. 2b).
3. w_1 and w_2 are coupled through the off-diagonal terms in B. The coupling causes 'leakage' of stress from the



(a) (b)

FIG. 2: The full numerical solutions for the stresses in a semi-infinite plane, when P is given by (14) and a narrow Gaussian-shaped σ_{xy} is applied to the boundary. Contour colors give stress magnitudes. (a) For $\epsilon_{ij} = 0.1$ the stress paths deviate only slightly from straight and the stresses along them hardly attenuate or leak. (b) For $\epsilon_{ij} = 0.5$ the cone of influence is still well defined, but both attenuation and leakage increase significantly. Due to the coupling, the stress at any point within the cone, e.g. A, is determined by the two sections of the main stress paths that extend between the source and points B_1 and B_2, respectively.

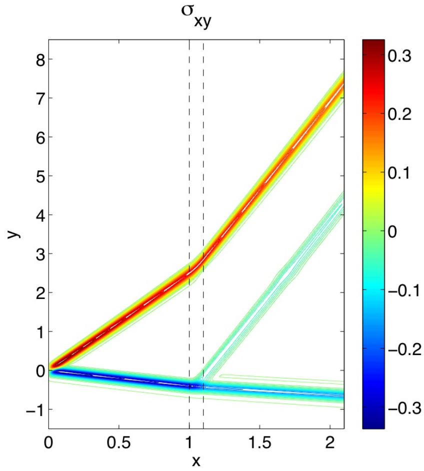


FIG. 3: Applying a a narrow Gaussian σ_{xx} at the boundary point $(0, 0)$ of a system with a strong gradient in P within a thin mid-layer between the dashed lines, sandwiched between two regions of (different) constant P, results in branching of the stress paths. The strength of a branch depends on the gradients in P and the characteristic variables along the branch. In this case, only the lower path leads to a noticeable branch. The colored contours give the magnitude of σ_{xy}

main characteristics into the cone of influence via secondary characteristics. Note that there can be no leakage to the outside of the cone since characteristics traced back from points outside the cone can neither connect to the load nor can they cross the cone boundaries. This conclusion is readily extended to the superposition of multiple boundary sources. The larger the gradients of P the stronger the coupling between the w_i. This is seen in figs. 2, where both leakage and attenuation increase from 2a to 2b.

To understand better the leakage phenomenon, let A within the cone of influence be the crossing point of two secondary characteristics, originating at points B_1 and B_2 along the main characteristics (fig. 2b). Stresses leak from B_i to A via the secondary characteristics, but along the way these pick up contributions from other secondary characteristics that they cross. Thus, the stress at A is determined by the stresses along the entire main characteristics segments between the source and the points B_i.

4. Large and localized gradients of P concentrate leakage, giving rise to 'branches' into the cone of influence. This phenomenon is illustrated in the full numerical solution shown in fig. 3, where a stress path crosses a large gradient in P, whereupon a new path branches off. The dependence of this phenomenon on local parameters is given in the caption.

Indeed, non-straight stress chains, coning, leakage and branching have been observed in experiments [4, 5, 20]. This supports our results, but only tentatively since most existing data is for systems that are not ideally isostatic, which introduces deviations from the above results, as discussed elsewhere [9].

To conclude, we have studied stress transmission in two-dimensional disordered isostatic granular materials both analytically and numerically. We have shown that the field equations for isostatic systems lead to a richer behavior than previously suspected. For the first time, leakage and branching of force chains can be derived explicitly and predicted from the isostaticity equations. Using a general stress-structure relation derived in the literature, we have reformulated the field equations as a particularly convenient system of hyperbolic equations. This made possible to derive explicitly several results for disordered systems. (i) Localized loads propagate predominantly along two meandering paths that make cone-like regions of influence. (ii) Stresses attenuate along the main paths. (iii) Stresses leak from the main paths into the cone of influence. (iv) Concentrated leakage leads to stress branching into the cone. (v) Stresses originating from different sources superpose and therefore cones of influence do not 'scatter' from one another. Eqs. (1)-(2) make it possible to derive the directions of chains, secondary chains and branches directly without the need to impose further conditions. These improve on heuristic isostatic models that impose conditions on the principal stress axes, and on models of ad-hoc chain splitting to explain branching [19].

Our reformulation of the field equations (12) is very convenient: it enables a general quantitative analysis of the stresses in disordered granular materials and allows explicit calculations of all the above phenomena because the terms governing attenuation and leakage/branching can be readily identified. This approach applies to any

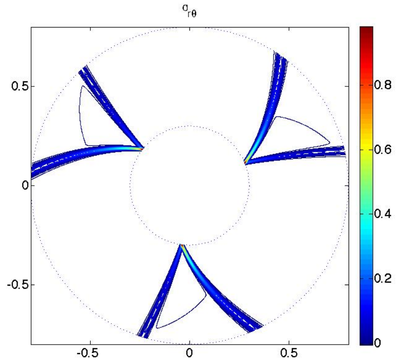


FIG. 4: An example of a stress solution, developing in an annulus of constant fabric tensor under shear. Three localized load sources of $\sigma_{r\theta}$ are applied along the inner boundary, giving rise to three pairs of stress paths, each forming a region of influence that resembles the cones of the semi-infinite plane. The colored contours give the magnitude of the displayed stresses.

variable fabric parameters regardless how large the disorder. The representation as a linear hyperbolic system is also convenient for numerical computations as standard solvers can be applied.

It is straightforward to extend our analysis for the semi-infinite plane to cylindrical geometries, which are relevant to a number of experiments in the literature [20, 21]. Indeed, we have derived the characteristic equations, analogous to (12), in cylindrical coordinates and have carried out some preliminary analysis. Fig. 4 shows an example of a solution of the stress that develops in an annulus under shear. We have also observed leakage and branching, as in rectangular geometries, and a more detailed analysis of cylindrical systems will be reported elsewhere [22].

Our results can be tested directly by applying localized loads to assemblies of photoelastic grains, computing from the structure fabric tensors, deriving stress solutions, and comparing to visualized force fields.

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