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ORIGINAL PAPER

Plug flow formation and growth in Da Vinci fluids

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Abstract A model is discussed for flow of dense granular matter — a da Vinci fluid. The local properties of the fluid are generically different from ordinary fluids in that energy is dissipated by solid friction. We discuss the equation of flow of such a fluid and show that it gives rise to formation and growth of plug regions — a phenomenon observed frequently in flow of granular matter. Simple explicit examples are presented to illustrate the evolution of plug flow regions.

Keywords Da Vinci fluid · Dense granular flow · Plug formation · Plug flow · Flow equations · Rheology

On length scales much larger than grain size, dry sand appears to flow similarly to ordinary fluids. Apriori, it should be possible to construct continuum flow equations for dense non-cohesive granular materials and indeed such an approach has a long history. The kinetic theory of so-called dense gases, which takes into considerations dissipation during collision between particles, presents a useful attempt of modeling granular flow. Unfortunately, it is only useful for low-concentration granular systems in very rapid flow [1–4]. For dense flows, when many particles rub against one another simultaneously, not only that formalism breaks down but also the concept of collision is not useful. This poses a serious problem since dense flows are ubiquitous in natural processes and in the industry, where most materials go through a particulate form in their processing. Examples include: processing of pharmaceutical powders and materials, agricultural grain transport, sand mobilization, and initiation of avalanches and landslides. It is the latter application in the field of geomaterials that attracted the attention of I. Vardoulakis [5].

In spite of the importance of modeling of dense granular flows, the development of a continuum description has been slow and fraught with difficulties [6]. A comprehensive review of the emerging understanding of dense granular flow has been given in [7].

We construct a model of dense granular flow, a da Vinci Fluid model, based on a microscopic picture of inter-granular interactions. We start from contact forces between grain, which can be decomposed into normal and solid friction forces, and discuss the stress tensor in regions of flow, where the strain rate tensor does not vanish identically. We call these plug free regions (PFRs). In contrast, in plug regions (PRs), the fluid moves as a rigid body and the strain rate tensor is identically zero. Plug formation is a well-known phenomenon in dense granular flow [8,9]. A reminiscent clustering instability appears in granular gases [10], although originating from a different physical mechanism. Formation and growth of plug regions is essential to the full description of dense granular flow, but including both PRs and PFRs in one model has proved difficult. We present a model that achieves this goal. We also show that nucleation and growth of plugs occurs around stationary points in the velocity profile. We further show that, shortly after formation, a plug size grows proportionally to $t^{1/3}$.

Consider the difference between ordinary and dense granular fluids. Conceptually, the repulsive forces between grains are equivalent to interactions between molecules in ordinary fluids. The main difference is that the friction

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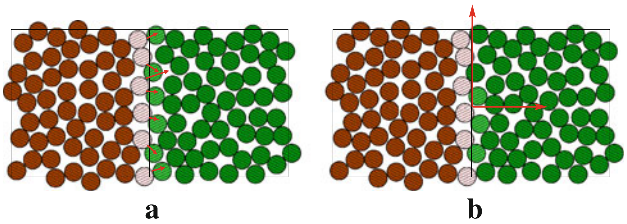

Fig. 1 **a** A sketch of two neighbouring volume elements of the Fluid share a layer surface, across which grains belonging to the different elements are in contact and transmit inter-granular forces; **b** The transmit inter-granular forces are coarse-grained into a normal and tangential force between the volume elements

between grains gives rise to non-central transmit inter-granular forces that cannot be described by two-body potentials[11]. Solid friction, described already by da Vinci [12], Amontons [13] and Coulomb [14], is an important energy-dissipating mechanism that transforms mechanical energy into heat, stored in intra-granular degrees of freedom. This is in contrast with the physics of ordinary viscous fluids. For example, the conventional dissipative term in the Navier-Stokes equation, which also represents conversion of mechanical energy into heat, only feeds energy from macroscopic fluid disturbances to fluctuations on much smaller scales. The viscosity term in the Navier-Stokes equation can be regarded a result of drag forces on individual molecules due to collisions, leading to viscous dissipation linear in the velocity spatial gradients.

To take account of grain-scale interactions and obtain a coarse-grained continuum description, consider a pair of neighbouring small volume elements in the fluid. These interact through their common interface (see Fig. 1). The force that one element applies on another, written in terms of total force density, consists of two terms: a pressure gradient $-\nabla P$ and a dissipative term. We assume that the dissipative term results only from inter-granular solid friction. We call such a fluid a da Vinci fluid (dVF) after Leonardo da Vinci , who conducted the first recorded experiments on solid friction [12]. This suggests a macroscopic effective friction coefficient that originates from both the inter granular friction coefficient and from the randomness of the local inter-granular interactions, as has been demonstrated in [15].

Schaeffer [16] seems to be the first to suggest a form of the stress tensor for granular fluids that takes into account inter-granular solid friction forces,

$$\sigma_{ij} = P\delta_{ij} - \mu P \frac{\tilde{T}_{ij}}{|\tilde{T}|} , \quad (1)$$

where P is the pressure and $\tilde{T}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the symmetric strain rate tensor, $|\tilde{T}|$ is its norm and μ is a dimensionless material-dependent constant. Based on experimental observations, Jop et al. [17] postulated a general dependence of μ on the inertial number I,

$$I = |\tilde{T}_{ij}| d \sqrt{\frac{\rho_s}{P}} , \quad (2)$$

where d is the grain size and ρ_s, its specific mass.

It should be emphasized that in the formulations of Schaeffer and of Jop et al. (Eq. (1)) the stress tensor applies only in PFRs. Technically, this is due to the term $\tilde{T}_{ij}/|\tilde{T}|$, which is undefined in PRs, where $|\tilde{T}|$ vanishes. The physical reason is that, in such regions, the threshold value given by the da Vinci-Amomtons- Coulomb law is not reached. PRs cannot be ignored, especially since, as we show below, non-uniform velocity fields are unstable to their formation and growth under dense flow conditions.

Here we go beyond the works of Schaeffer and Jop et al. and we formulate a complete theory that describes the flow and dynamics of both PRs and PFRs with one set of equations. In passing, we mention that we have derived theoretically the stress tensor in PFRs by careful coarse-graining of the grain-level interactions. We have found that it is more involved than the one postulated previously (Eq. (1)), but that it has similar features. In particular, that the stress tensor is of order zero in the strain rate tensor. Since the main issue of this paper is the study of plug dynamics, we leave the details of this derivation to a later report [18]. In our model we introduce a scalar field ψ that not only describes the evolution of contours of PRs but it also augments the set of equations to describe fully the flow and dynamics of PFRs and PRs.

We show that it is possible to construct a well-defined stress tensor, made of two parts: one relating to the normal forces and another to the friction forces [18]:

$$\sigma_{ij} = \sigma_{ij}^{(n)} + \sigma_{ij}^{(f)} . \quad (3)$$

In this expression, the first term on the right hand side describes contribution of normal forces between volume elements and the second originates from solid friction between them. Note that the boundary between volume elements, which contain a large number of grains, is a conceptual construct and, therefore, the net force due to normal forces between grains across such a boundary need not necessarily be normal to it. Similarly, the net friction force between volume elements across a boundary need not necessarily align with the boundary.

It should be noted that, once a stress tensor has been defined, the flow dynamics are determined. This can be exemplified, using the form (1), proposed by Jop et al. [17], showing that it leads to an autonomous set of equations, as explained in the following.

The pressure must diverge at a finite maximal density ρ_c that is below that of close (likely even loose) random packing. We aim to model only dense flows of average density just below this value, $\delta\rho = \rho_c - \bar{\rho} << \rho_c$. A full description of the system is obtained from the following set of equations.

(i) An equation of state, $P = P(\rho)$; (ii) The equation of continuity

$$\partial_t \rho + \nabla \cdot [\rho \mathbf{v}] = 0; \tag{4}$$

(iii) Newton's equation of motion,

$$\rho \left[\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}\right] = -\nabla \cdot \sigma + \rho \mathbf{g}. \tag{5}$$

Here $\mathbf{g}$ is a local external force density acting on the system, e.g. due to stirring, interaction with the boundary or gravity. Equations (4) and (5) are the ones that evolve the density and velocity profiles, but they are valid only wherever $\nabla \mathbf{v}$ does not vanish. In ordinary liquids, it is often possible to obviate the equation of state by the simplifying assumption of incompressibility, but whether or not this is a good approximation for dense granular fluids is an open question.

It is evident that the acceleration field within a PR is spatially uniform; it can be determined from the total force on the region divided by its mass. It is important to note that changes in the position of the boundaries of the PR must be accompanied by internal redistribution of stresses. We assume here that the response of the stress field is much faster than any other process in the system and can be considered practically instantaneous.

The total force on a plug, occupying a region Ω, due to the normal forces applied on the region boundary $\partial\Omega$ by the adjacent fluid, is

$$\mathbf{F_P^n} = -\oint_{\partial\Omega} \sigma_+^{(n)}(s) \cdot \hat{\mathbf{n}}(s) ds, \tag{6}$$

where ds is an infinitesimal surface element on $\partial\Omega$ and $\hat{n}(s)$ is an outward pointing unit vector normal to the boundary at point s. $\sigma_+^{(n)}$ is the stress due to inter-granular normal forces just at the outer side of the boundary $\partial\Omega$. The friction force acting on the region is correspondingly given by

$$\mathbf{F_P^f} = -\mu \oint_{\partial\Omega} \sigma_+^{(\mathbf{f})}(\mathbf{s}) \cdot \mathbf{n}(\mathbf{s}) \mathbf{ds}\ , \tag{7}$$

where $\sigma_+^{(f)}$ is the frictional stress at the outer side of the PR boundary. The total mass of the PR is

$$M_P = \int_{\Omega} \rho(\mathbf{r}) \mathbf{d^3 r} \tag{8}$$

and, since the term $\mathbf{v} \cdot \nabla \mathbf{v}$ vanishes in Ω, the PR acceleration is

$$\frac{\partial \mathbf{v_P}}{\partial t} = \frac{\mathbf{F_P^n} + \mathbf{F_P^f}}{M_P}. \tag{9}$$

We demonstrate below that PRs are unstable and discuss how they grow. Specifically, to understand the flow behavior of dVF, we need to consider the kinematics of PR boundaries. To this end, we define the scalar function

$$\psi = \frac{1}{2} Tr\left(\tilde{T}^2\right), \tag{10}$$

which is finite only outside PRs. The growth of a PR is equivalent to an expansion of the external contour lines of zero ψ surrounding it. We, therefore, focus on the kinematics of these contours. The equation of motion for the scalar field ψ is

$$\frac{\partial \psi}{\partial t} + \mathbf{V}_\psi \cdot \nabla \psi = \mathbf{0}, \tag{11}$$

where $V_\psi(\mathbf{r})$ is the velocity of the contour line of $\psi(\mathbf{r})$ at location $\mathbf{r}$. Evidently, only the component of V_ψ normal to $\partial\Omega$ is necessary to describe the expansion of the PR. This component is given by

$$\mathbf{V_\psi^n} = -\frac{\nabla \psi}{|\nabla \psi|^2} \frac{\partial \psi}{\partial \mathbf{t}}. \tag{12}$$

Note that, in expression (12), $\nabla\psi/|\nabla\psi|$ is a unit vector pointing outwards in the direction normal to the constant contour of ψ. It is straightforward to see, by direct substitution, that this expression is nothing but the formal solution to (11). To make use of (12) and obtain an explicit expression for the evolution of the plug boundary, one first uses Eq. (10) to rewrite the right hand side in terms of the velocity field outside the plug and then one uses the equation of motion (5) to write the right hand side directly in terms of the fluid velocity field at the boundary. With $\mathbf{V_\psi^n}$ given, we can determine the evolution of any constant contour of ψ and, in particular, that of the contour $\psi = 0$ just at the plug boundary.

The gradients on the right hand side of Eq. (12) should be taken across the boundary of the PR and it may be discontinuous. Nevertheless, a scrutiny of the numerator and the denominator will convince the reader that these discontinuities and the corresponding δ-functions cancel out and leave a well-behaved term. This is also illustrated in the examples discussed next.

To gain insight into the dynamics of PR growth in dVF, we consider now an example that makes possible an explicit solution. Let a dVF be confined between two far-away boundaries at $x = \pm L$ and between $z = \pm\infty$ in the z-direction (Fig. 2a). We postulate an initial uniform density $\rho(x, t = 0) = \rho_0$ and an arbitrary initial velocity profile in the z-direction, $v_0(x)\hat{z} = v(x, t = 0)\hat{z}$. Compressive forces in the x-direction are applied uniformly across the boundaries at $x = \pm L$. These forces give rise to a stress tensor with a non-zero component σ_{xx}, taken to be constant. We also postulate zero strain at these boundaries. This condition leads to a shear stress on the boundaries $\sigma_{xz} = \sigma_{zx} = \mu\sigma_{xx}$. Due to the symmetry of the system and the boundary loading, neither the forces nor the initial conditions can change the density distribution, which alleviates the need to solve for the equation of state.

Consider first an initial continuous and monotonic velocity profile $v_0(x)$, i.e. $\partial v_0(x)/\partial x \neq 0$ for all x. Then Eq. (5)

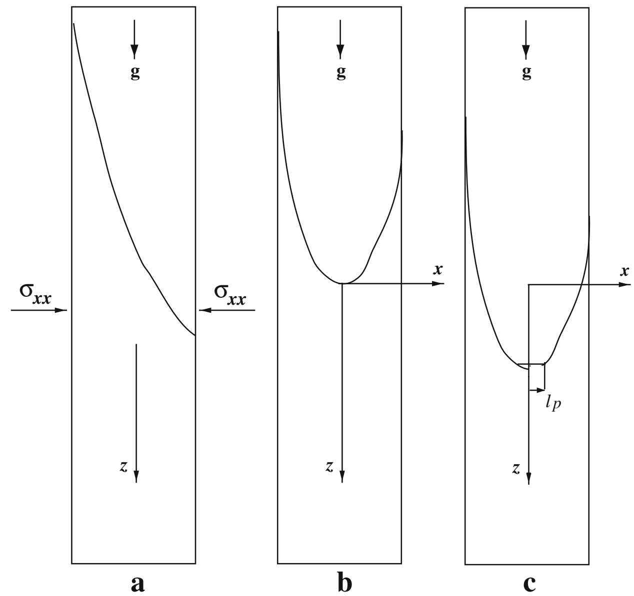

Fig. 2 **a** A simple uni-directional flow in the z-direction under gravity g. The initial downward velocity profile increases from *left to right*. **b** Same as in a but the velocity profile has a local maximum. **c** Nucleation and growth of a plug around a local maximum of the velocity profile. Initially, all the fluid accelerates at g except for the streamline through the maximum, which is slowed down from both sides by friction forces. Consequently, a plug forms and grows around this point

reduces to $\partial v/\partial t = g$, whose solution is simple: the velocity profile remains constant with time, $v(x,t) = v(x,t=0) + gt$.

Next, consider an initial velocity profile containing a local maximum, at $x = 0$ (Fig. 2b). A PR nucleates at the maximum, as described in detail in [19]. Basically, this is because the streamline at $x = 0$ experiences friction forces opposing the flow from both sides and it must decelerate relative to its surrounding fluid. Eventually, its velocity matches that of a neighbour streamline and thenceforth they move together, forming a nuclear plug. The PR is slowed down by its surrounding fluid and continues growing by the same mechanism.

We wish to study the growth rate of the PR around the maximum. In regions not yet reached by the PR boundary, the velocity profile is monotonic and, as discussed above (see also [19]), it does not change with time. Thus, for sufficiently short time t the growth of the PR is dominated by the velocity profile near the maximum. For illustration, let us suppose that the velocity profile can be expanded in series around the maximum as

$$v(x,t=0) = U - \alpha\,(x/x_0)^2 + O\left[(x/x_0)^3\right], \tag{13}$$

where α and x_0 have units of velocity and length, respectively. The velocity profile at a later time t is given by

$$v(x,t) = [v(x,t=0) + gt]\,\theta\left(t - t_p(x)\right) + v_P(t)\theta\left(t_p(x) - t\right), \tag{14}$$

where θ is the Heavyside step function, t_p is the time when the plug boundary reaches point x and $v_p(t)$ is the velocity of the PR at time t. The location of the PR boundary at time t is $l_p(t)$ (Fig. 2c) and the acceleration of the PR is

$$\frac{\partial v_p}{\partial t} = g - \frac{\mu\sigma_{xx}}{\rho_0 l_p(t)}. \tag{15}$$

The last term on the right of (15) represents deceleration due to friction on the PR boundaries. The expansion rate of the PR can be found directly from (12),

$$V_\Psi = \frac{\partial l_p}{\partial t} = \frac{\mu\sigma_{xx}x_0^2}{2\rho_0 v_0 l_p^2}. \tag{16}$$

Alternatively, (16) can be obtained by solving for the time δt that it takes a streamline a distance δx away from the boundary to match velocity with the PR. From (16) we now obtain that the PR boundary grows as

$$l_p(t) = \left[3\frac{\mu\sigma_{xx}x_0^2}{2\rho_0 v_0}t\right]^{1/3}. \tag{17}$$

Equations (14), (15) and (17) provide a full solution for the velocity profile. The expansion rate of PRs as $t^{1/3}$, described by (17), is generic and can be understood as follows. The friction force is proportional to the mean normal stress times the surface area of the PR boundary, while the mass is proportional to its volume. In our example of gravity flow, the PR experiences a constant friction force from its surrounding material and its mass increases proportionally to $l_p(t)$ as it expands. Consequently, the acceleration decreases inversely proportional to $l_p(t)$. Similarly, given an initial velocity profile $v_0(x,y,t=0)$ that is cylindrically symmetric around a local maximum, the friction per unit height experienced by a PR of height h is proportional to $2\pi l_p(t)$, while its mass per unit height increases as πl_p^2. The reduction in acceleration is then also inversely proportional to l_p, leading again to the result that l_p grows as $t^{1/3}$. It can be shown that the linear size of the region (however defined) increases as $t^{1/3}$ for any velocity profile that can be expanded around a local maximum at (x_0, y_0) as

$$v(x-x_0, y-y_0) = U - \alpha\left(\frac{(x-x_0)^2}{R_1^2} + \frac{(y-y_0)^2}{R_2^2}\right) + \ldots \tag{18}$$

To conclude, we have developed the flow equations of a fluid dominated by da Vinci - Amontons - Coulomb solid friction. In contrast to existing models, our flow equations are valid for arbitrary flow regions, whether plug (i.e. of uniform velocity) or not. This was done by introducing into the set of equations an equation for the evolution of a scalar field

ψ that couples to the strain rate and describes the evolution of contours of PRs. Furthermore, a key advantage of this model is that it gives rise naturally to unstable flows in the sense that the flow equations lead to formation and growth of plug regions. This unique feature of dense granular flow is extremely important but hitherto hardly studied. Our flow model is valid beyond these plug nucleation instabilities and it describes the formation, expansion and motion of the plugs region alongside the kinematics of regions of nonuniform flow.

We have discussed simple cases that are amenable to analytic treatment and we have shown that a generic feature of the flow is that, once a plug region has formed, its linear size increases with time t as $t^{1/3}$. It is interesting that this growth law of PRs, which we have found in a model for dense granular flows, has been observed also in simulations of much more dilute granular gases [20,21].

This minimal model appears to capture well key features of slow flow of dense granular materials, when grains maintain significant contact at all time. This is in spite of the simplifications that we have employed. In particular, in our examples we presumed, for illustration purposes, that the friction coefficient, μ is uniform in space. It has been pointed out by Jop et al. [17] and later workers [22] that the friction coefficient depends in fact on the inertial number I (see expression (2)), and therefore on the flow pattern. Our equations can be extended straightforwardly to accommodate this observation and we will report the results of this extension elsewhere.

Experimental and numerical work, which we intend to take up in the future, is still needed to put this model to the test. In particular, it would be interesting to test the model's prediction for the growth rate of plug regions.

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