Structural Evolution of Granular Systems: Theory
Supplementary Calculation

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Proof: the fractions Q_i are normalised to first order

It is claimed in [1] that $\sum_k \delta Q_k = 0$, which implies that the cell fractions Q_k remain normalised to first order when expanded close to steady state. Close to the steady state, the full linear expansion of the evolution equations is:

$$\begin{aligned}
\delta \dot{Q}_k = \quad & \frac{1}{2}\sum_{i=3}^{k-1}\left[p_{i,k-i+2}\left(Q^s_{k-i+2}\delta Q_i + Q^s_i \delta Q_{k-i+2}\right) - q_{k,i}\delta Q_k\right](1+\delta_{i,k-i+2}) \\
& - \sum_{i=k+1}^{i_\mathrm{m}}\left[p_{k,i-k+2}(Q^s_k \delta Q_k + Q^s_{i-k+2}\delta Q_{i+k-2}) - q_{i,k}\delta Q_i\right](1+\delta_{i,k-i+2}) \\
- Q^s_k \sum_{\ell=3}^{\mathcal{C}} & \left\{ \frac{1}{2}\sum_{i=3}^{\ell-1}\left[p_{i,\ell-i+2}\left(Q^s_{\ell-i+2}\delta Q_i + Q^s_i \delta Q_{\ell-i+2}\right) - q_{\ell,i}\delta Q_\ell\right](1+\delta_{i,\ell-i+2}) \right. \\
& \left. - \sum_{i=\ell+1}^{i_\mathrm{m}}\left[p_{\ell,i-\ell+2}(Q^s_\ell \delta Q_\ell + Q^s_{i-\ell+2}\delta Q_{i+\ell-2}) - q_{i,\ell}\delta Q_i\right](1+\delta_{i,\ell-i+2}) \right\} ,
\end{aligned}$$

with $i_m \equiv \left|\left(\frac{\mathcal{C}+2}{2}\right)\right|$. Summing over the index k and using $\sum_k Q_k = 1$, we obtain

$$\begin{aligned}
\sum_{k=3}^{\mathcal{C}} \delta \dot{Q}_k = \quad & \sum_{k=3}^{\mathcal{C}} \left\{ \frac{1}{2}\sum_{i=3}^{k-1}\left[p_{i,k-i+2}\left(Q^s_{k-i+2}\delta Q_i + Q^s_i \delta Q_{k-i+2}\right) - q_{k,i}\delta Q_k\right](1+\delta_{i,k-i+2}) \right. \\
& \left. - \sum_{i=k+1}^{i_\mathrm{m}}\left[p_{k,i-k+2}(Q^s_k \delta Q_k + Q^s_{i-k+2}\delta Q_{i+k-2}) - q_{i,k}\delta Q_i\right](1+\delta_{i,k-i+2}) \right\} \\
- \underbrace{\sum_k Q^s_k}_{=1} \sum_{\ell=3}^{\mathcal{C}} & \left\{ \frac{1}{2}\sum_{i=3}^{\ell-1}\left[p_{i,\ell-i+2}\left(Q^s_{\ell-i+2}\delta Q_i + Q^s_i \delta Q_{\ell-i+2}\right) - q_{\ell,i}\delta Q_\ell\right](1+\delta_{i,\ell-i+2}) \right. \\
& \left. - \sum_{i=\ell+1}^{i_\mathrm{m}}\left[p_{\ell,i-\ell+2}(Q^s_\ell \delta Q_\ell + Q^s_{i-\ell+2}\delta Q_{i+\ell-2}) - q_{i,\ell}\delta Q_i\right](1+\delta_{i,\ell-i+2}) \right\} .
\end{aligned}$$

Renaming the index ℓ as k, the individual sums neatly cancel out:

$$\begin{aligned}
\sum_{k=3}^{\mathcal{C}} \delta \dot{Q}_k = & \sum_{k=3}^{\mathcal{C}} \left\{ \frac{1}{2}\sum_{i=3}^{k-1}\left[p_{i,k-i+2}\left(Q^s_{k-i+2}\delta Q_i + Q^s_i \delta Q_{k-i+2}\right) - q_{k,i}\delta Q_k\right](1+\delta_{i,k-i+2}) \right. \\
& \left. - \sum_{i=k+1}^{i_\mathrm{m}}\left[p_{k,i-k+2}(Q^s_k \delta Q_k + Q^s_{i-k+2}\delta Q_{i+k-2}) - q_{i,k}\delta Q_i\right](1+\delta_{i,k-i+2}) \right\} \\
& - \sum_{k=3}^{\mathcal{C}} \left\{ \frac{1}{2}\sum_{i=3}^{k-1}\left[p_{i,k-i+2}\left(Q^s_{k-i+2}\delta Q_i + Q^s_i \delta Q_{k-i+2}\right) - q_{k,i}\delta Q_k\right](1+\delta_{i,k-i+2}) \right. \\
& \left. - \sum_{i=k+1}^{i_\mathrm{m}}\left[p_{k,i-k+2}(Q^s_k \delta Q_k + Q^s_{i-k+2}\delta Q_{i+k-2}) - q_{i,k}\delta Q_i\right](1+\delta_{i,k-i+2}) \right\} \\
& = 0. \qquad (1)
\end{aligned}$$

It follows that $\sum_{k=3}^{\mathcal{C}} \delta Q_k$ is constant.
Since $\sum_{k=3}^{\mathcal{C}} Q_k = 1$ by definition at all times then

$$\sum_{k=3}^{\mathcal{C}} \delta Q_k(t) = 0 \tag{2}$$

at all times and, in particular, in the linearised regime close to the steady state.

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[1] C. Wanjura, P. Gago, T. Matsushima, R. Blumenfeld (2019), preprint available at https://arxiv.org/abs/1904.06549.