Mesoscale fracture propagation: Noise-free and noise-driven steady states, periodicity and intermittency

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Abstract

This paper generalizes a recently proposed model [1] for the mesoscale dynamics of mode I crack propagation in brittle non-crystalline materials. Using the observed slow stress relaxation rate at the crack front, a first-principles dynamic equation is constructed, based on the Yoffe solution. Further postulating a generic relation between the tip velocity and the local stress, this equation is solved and a detailed analysis of its solutions is presented. Within an effective continuum approach, the nature of the solution depends on only one material-dependent parameter, λ. In the absence of noise, the propagation is either at a steady state or the velocity oscillates periodically in time. The model is extended to describe fracture propagation under mode III loading. Effects of noise are discussed in detail next. It is found that noise can strongly modify the propagation dynamics and may give rise to: intermittent propagation, quasi-periodic behavior, and a novel continuous spectrum of noise-driven steady states. A phase diagram of the propagation state in the parameter space of noise amplitude and frequency is proposed.

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1. Introduction

The dynamics of crack propagation have been a long-standing problem in materials science, and the quest for ever better materials keeps driving an extensive effort to understand this processes. Fracturing abounds in natural and man-made materials and is therefore a worthwhile problem both theoretically and technologically. All too often, a fracture becomes unstable and turns catastrophic in the sense that the crack starts running and leads to an immediate material failure. This feature of finality, and the attempt to control the failure mechanism, have led to an enormous effort by engineers and scientists. Yet, in spite of the large body of work, quite a few fundamental issues have stubbornly resisted resolution. For example, many observations made on quasi-two-dimensional systems indicate that running cracks reach a limiting steady-state propagation velocity of about 40% of the Rayleigh wave speed. Predictions of the limiting velocity date back to Sir N. F. Mott [2], who used Griffith's energy balance argument [3], to propose the existence of such a velocity. This terminal velocity was later [4] proposed to be the RWS, which is the speed of propagation of surface waves along the (idealized) crack surfaces that are left behind the crack front. The reason why these predictions are higher by a factor of two or so than real measurements is not explained within the paradigm of energy balance considerations. Another poorly understood issue concerns the physical mechanisms for crack initiation and arrest and the reason for the stress hysteresis between the two [5][6]. While the phenomenon of lattice trapping [7] seems to be related to this issue, and an initial understanding of this phenomenon starts to emerge from atomic models [8], the hysteresis remains largely unexplained on mesoscopic and macroscopic scales. Another piece in the puzzle, which was highlighted a few years, is the origin of the observed periodic-like oscillations of the propagation velocity in amorphous polymeric materials such as PMMA [9][10]. The relation between these oscillations and the properties of the material is far from clear. Another important problem, which has been strongly emphasized recently, is the relation between material properties and the roughness of the fracture surfaces that are left behind. While it is obvious that there is a close connection between the two, there are very few attempts to pinpoint this connection by a suitable theory.

As a consequance of the above problems (and other that have not been listed), there is currently no general consensus on the form of the equations of motion that govern continuum fracture dynamics. Vis-á-vis this state of affairs on the theoretical front, there exists a wealth of experimental data on many aspects of the fracturing process in many materials, including, the speeds of steady state propagation, the stress intensity factor at the propagating tip, the roughness of the fracture surfaces, the toughness of the materials and so on. Some of these data are deduced from quite elaborate measurements whose interpretation is not completely certain. Nevertheless, the plethora of information makes it possible to test proposed models against existing evidence.

The long history of this problem contains many approaches that have been employed to obtain predictive models. Whether explicitly or implicitly, the currently reigning approaches use traditional quasi-static and/or near-equilibrium concepts to determine the dynamics of the propagation process. While the resulting models led to much progress, they failed so far to satisfactory resolve the problems listed above. Several reasons are responsible for that: First, analysis of fast fracture processes is usually restricted to post-mortem measurements of the fractured system, while the process itself is too fast to capture. It is only in recent years that experiments emerged where the dynamic process can be monitored on the run. Second, and in this author's opinion probably the more important obstacle, is the fact that the fracture phenomenon is governed by *different physical mechanisms on different length-scales.* This aspect has not received the attention that it deserves and is worth some elaboration. In ideally brittle propagation, the crack tip is atomically sharp, which implies that atomic potentials are significant. This points to the relevance of the physics on scales of $5 - 10\AA$. Realistic potentials (even assuming that simple pair potentials suffice to describe the behavior of the material on this scale) are usually anharmonic at the pertinent strains near the crack tip, giving rise to strong nonlinear effects that influence the propagation. For large lengthscales ($> \mu$m for pure materials and $> 100\mu$m for polycrystalline materials), continuum linear elasticity seems to describe the stress field and the far-away elastic energetics quite well. This is exactly because fractures grow slower than any of the bulk speeds of sound. Since the bulk stress relaxes to its static value at the speed of sound, which is faster than the crack's propagation rate (which of the sound speeds is of no importance because the crack propagates macroscopally slower than any of them), then the macroscopic system can be considered at a steady state in a frame of reference that moves with the crack tip. This is also the basis for using energetic and quasi-static approaches for the far field. Contour integral calculations of the influx of energy into the crack tip are a particular example of calculations that remain valid only as long as the contours are taken well away from the tip. However, between the atomic and the continuous scales there are at least two more relevant length-scales. One is that of the so-called cohesive zone. This is the region where the continuous stress field description breaks down due to the discreteness of the lattice, and the continuous stress-divergence needs to be modified due to atomic effects. This length scale is of the order of several lattice constants and is about one order of magnitude above the atomic scale, namely, $\sim 10 - 50\AA$. Another length-scale, and the one we focus on in this work, is the scale defined by the sizes of the microcracks that form dynamically in front of the propagating tip. The term microcracks is used here generically since the sizes involved can range from a few nanometers to many microns. For lack of a better term, this is called mesoscale here. It is noted though that if the material is granular as well, where grains can be of sizes from tens to hundreds of microns, then there will be yet a larger relevant scale, which is still shorter than the continuum. In the following we specialize to materials that contain no such grains. Since typical length scales in the mesoscale are larger than the cohesive zone, one could think that the continuous description should apply. While this is correct in principle, continuum theory is not easy to use in any straightforward fashion due to the strong textural disorder. The texture essentially translates into many small complicated boundaries that need to be taken into consideration when solving for the continuous stress and strain fields. This is clearly an impossible task even with today's computational powers and a mesoscale theory is needed, which can bridge between the atomic and continuum descriptions. An attempt to gloss over the disorder just ahead of the crack by naive coarse-graining does not work for the dynamical problem for the following reasons. Microcracks, which nucleate from lower-scale defects (e.g., vacancies or dislocations), can cover a region of order of microns in front of the tip. Such a region is indeed observed and is called (albeit in connection with plastic deformations rather than brittle propagation) the processing zone (PZ). It is claimed here that the macroscopically observed dynamics of crack propagation are dictated by the physics on this scale, both due to disorder-induced enhanced scattering of sound waves and due to strong fluctuations that are introduced by microcracks. In particular, it is suggested that on this 'dynamic scale' the stress field relaxes very slowly and the propagation is swifter than the *local* stress relaxation. This gives rise to a supersonic-like behavior in spite of the fact that macroscopically the crack moves well below the speed of sound. The dynamics in this regime are essentially the reason for the failure of energy balance considerations to account for the tip kinetics. It should be emphasized that the fact that the dynamics are dominated by the mesoscale does not invalidate the long range continuum energy calculation. These calculations, however, need to be applied not to the bare crack tip, but rather to the tip 'dressed' by the disordered zone that surrounds it.

In the continuum description, the stress field very close to the tip drops as an inverse square root (e.g., the Yoffe solution for mode I [11] or the inverse square root for mode III, see below). The angle-independent prefactor of the inverse square root is termed the stress intensity factor, K, and is defined through the relation

$$\sigma_{\alpha\beta} = K f_{\alpha\beta}(\theta)/\sqrt{2\pi r} \ , \tag{1.1}$$

where $\sigma_{\alpha\beta}$ is the stress tensor, r is the distance from the moving tip and $f_{\alpha\beta}$ is a tensor that depends only on the azimuthal angle θ. This tensor can be obtained using formal expansions in a complete set of basis functions, such as spherical harmonics. In traditional treatments, the stress intensity factor is presumed to be modified as the crack moves, with all dependence on time, speed of propagation, the cracking mode (I, II or III) and the history of the propagation being lumped into this parameter. It is argued here that this view may miss some important underlying physics, As we now proceed to explain. At the basis of the form (1.1), lies the assumption that the propagation is subsonic. In other words, it is usually taken for granted that, similarly to the far field case, the stress field near the tip relaxes to its static value sufficiently fast such that the stress field has essentially the *static* functional form in the tip's moving frame. Under this assumption the crack tip coincides with the singularity ($r \to 0$ in the continuous description) of the stress field. While the many observations of a limiting propagating speed that is well below the SWS, seem to support this conclusion for the far field, there is a leap of faith from the low crack propagation speed to the assumption that the far-field energetics dictate the crack dynamics on the mesoscale. A necessary condition for energy balance considerations to hold true is that the flow of information to the tip about the changing stress field is *locally* (not globally) faster than the propagation rate. Following existing experimental evidence [12], it was proposed in [1] that, in most cases, the tip is well shielded from the far field by the highly inhomogeneous processing zone. The shielding promotes mesoscale physical mechanisms that dominate near the Crack tip and make the process a strongly non-equilibrium one. Ultimately, it is the very non-equilibrium nature of the problem that invalidates bulk energetic considerations. Since energetic considerations have been the dominant paradigm for many decades, it is important to carefully reexamine this approach in order to substantiate the current claim. Cracking leads to formation of a new boundary in the continuous material. The information that such a boundary has formed travels ahead of the crack at the appropriate speed of sound (which we will take as the SWS in this case, following [1] and [8]). If this speed is higher than the crack propagation velocity the stress *at the tip* stays at its static value and quasi-static calculations hold true. But suppose that due to some local physical constraints (e.g., those discussed below) the tip can propagate at velocities that are higher than the local SWS. In this case, the tip will travel ahead of the wave front that relaxes the field and the stress near the tip is *lower* than the static value. It is proposed here that this scenario may apply in many cases, in spite of the observed global limiting crack propagation being well below the bulk SWS.

Indeed, it has already been suggested that the discrepancy between the predicted and measured terminal crack speeds should be traced to the microcracking and the disorder formation ahead of the crack [5][13]. The (nearly) first-principles theory presented here is an attempt to put meat on this suggestion. It is well established that the properties of the material change drastically near the tip, a phenomenon that is referred to, in some contexts, as the appearance of a PZ into which the crack propagates. The microcracks in the PZ scatter sound waves on a wide range of wavelengths and therefore suppress the local stress relaxation rate. Being strongly disordered, the speed of sound in the PZ will then be markedly lower than in the homogeneous material. The attenuation of the sound waves becomes more pronounced with increasing microcrack density and is therefore expected to be strongest very close to the tip. Evidence supporting this interpretation comes from the measurements of Ma and Freund [12], who found a significant reduction in the local stress relaxation rate near crack tips. Specifically, they have shown that the relaxation takes place over *many tens of μsecs* after the crack arrests. To appreciate the significance of this observation, one should note that if the relaxation rate were at the homogeneous SWS, then the stress field would have attained its static value about an order of magnitude faster [14]! Although this observation was interpreted differently at the time, it is suggested here that the heterogeneity-induced strong scattering in the PZ is the simplest explanation for the anomalously slow relaxation rate. Nevertheless, it should be emphasized that this interpretation is not essential to the analysis presented here. What is essential is the mere occurrence of such a slow stress relaxation, and it is this alone that gives rise to the results obtained below.

As idscussed, a locally supersonic propagation calls to question the aforementioned assumptions that underlie traditional treatments. In particular, it casts doubt on the utility of the stress intensity factor in Eq. (1.1). In the following we dispose of this quantity altogether since it is not necessarily the best parameter to monitor. This notwithstanding, it should be acknowledged that the stress intensity factor has been measured in overwhelmingly many experiments. Therefore, it would be constructive to interpret our results in terms of this quantity, if only to allow for experimental cross-checking of the present mesoscale theory.

Before proceeding to construct the theory it should be mentioned that a couple of dynamical models for crack propagation have been suggested in recent years to account for the reduced limiting speed [15]. These models treat the tip as a massive particle with inertia and dissipation and start from a phenomenological dynamic equation. It would be fair to say that the connection of such a phenomenological approach to the local physics that govern the dynamics on the length-scales of the PZ is far from clear. Here, we take a different route and derive, as much as we can from first principles, the equation of motion of the tip. Starting from the idea that the tip responds to the local stress field, and that the stress field relaxes slowly relative to the homogeneous bulk, a dynamic equation is obtained for the local stress evolution. Although nonlinear, this equation is not difficult to analyse in detail and even to solve formally. Furthermore, upon postulating a generic (but not specific) form of the local velocity-stress response, the analysis can be made explicit. All in all, this theory takes into acount the following experimental evidence: the aforementioned slow relaxation rates at the tip [12]; the stress hysteresis between crack initiation and arrest [5][6]; and, to obtain explicit results, the generic form of the velocity as a function of the local stress at the tip. The latter information can be found both from experimental measurements and from a recent first-principles calculation on atomic systems [8]. These ingredients alone turn out to give rise to a spectacularly rich behavior of the propagating crack even without considering noise.

Noise, however, is inherent to the fracturing process even at zero temperature, and should therefore be taken into account in any attempt to model the fracture process. Therefore, after establishing the theory, we will turn to analyse effects of noise in great detail. Noise in this system can originate first from the vibrations that are excited in the atomic system due to the violent atomic bond breaking events, and second, from the distribution of the microcracks in the PZ. It will be shown that noise can strongly modify the dynamics. More specifically, depending only on one dimensionless material parameter, λ, and the noise characteristics, the crack tip can display several intriguing modes of behavior: Steady propagation at a constant limiting velocity; periodically oscillating propagation; quasi-periodic propagation; intermittent propagation; or a noise-driven steady state with a spectrum of average limiting speeds.

The present approach allows to separate between two regimes: When the microcracks in the PZ are sufficiently small and of frequent occurrence they give rise to fast fluctuations that average out in experimental measurements. In this case, the tip experiences effectively a locally continuous medium with mechanical properties that are different from the bulk. For this regime, we construct an effective continuum theory, which turns out to yield either a steady state or periodic oscillations in the propagation speed, depending on λ. The steady state limiting velocity is shown to be a stable fixed point of the dynamic equation. In media where the local SWS is lower than some critical value (to be defined below) no steady state is possible and the propagation is perfectly periodic.

After analysing the ideal noise-free propagation the paper proceeds to study effects of noise. By the term noise we refer to fluctuations that are sufficiently large to be noticeable on timescales that are relevant to measurements, but which are not too large so as to invalidate the effective continuum approach. When microcracks in the PZ are too large, a second level of modification is needed in the theory. In this case the dominant growth mechanism crosses over to what is probably best termed a 'microcrack association' mode, wherein the crack propagates continuously and associates the large microcracks as these are encountered along its path. The growth dynamics in this regime are sensitive to the exact spatial and Size distributions of the microcracks. This case deserves a detailed statistical analysis and will only be mentioned briefly here. A more detailed study of large amplitude noise will be addressed elsewhere [16].

2. The equation of motion

As the crack propagates the stress field adjusts to the new boundary that forms through sound waves that travel in the medium. Within the traditional picture the stress field relaxes at the speed of the Rayleigh waves that travel along the newly formed surfaces. This picture, though, is difficult to reconcile with the observations that the surfaces that are left behind are rough on many lengthscales. Such roughness strongly interferes with a free propagation of the surface waves and is bound to lead to nonlinear effects that can range from localization to nonlinear vibrational excitations on a large range of scales (Moreover, the interaction of the surface waves with the vibrating medium is hopelessly complicated to analyse, an aspect that deserves a thorough investigation in its own right). Having discarded the Rayleigh wave speed as the pertinent relaxation rate in this regime, one may wonder which sound speed is relevant? This question is immaterial for the purpose of the theory developed here. The only important point is that there exists such a local relaxation rate. Whether it is the shear wave speed, as argued by Holian et al. [8] on the atomic scale, or a different rate, is not entirely clear. For concreteness, we shall refer to this quantity as the local stress relaxation rate (LSRR). Due to the inhomogeneity that forms ahead of the crack, the stress relaxation rate in front of the tip, c, is lower than the bulk value by a factor of 2-3, as found by Ma and Freund [12]. It will be assumed initially that c is constant and independent of the degree of inhomogeneity. This assumption, which simplifies the derivation of the equation of motion, is not essential and will be lifted later. To derive the dynamic equation for mode I, we start from the expression for the stress field just ahead of the moving crack, as was derived by Yoffe [17],

$$\sigma = \sigma_\infty \left[(\zeta + a)/\sqrt{\zeta(\zeta + 2a)} \right] . \tag{2.1}$$

In this expression, ζ is the distance from the tip, σ_∞ is the mode I tensile stress applied perpendicular to the propagation axis far away from the crack, and a is the crack's length. In what follows the stress will be measured in units of σ_∞ and we will use the dimensionless stress $\sigma \to \sigma/\sigma_\infty (> 1)$. The Yoffe solution assumes implicitly that the singularity of the stress field occurs at the tip, $\zeta = 0$, and increases as $\sim 1/\sqrt{\zeta}$ as the crack tip is approached, $\zeta \ll a$. The singularity is cut off, of course, on the scale of the cohesive zone ($\sim 10 - 50 \mathring{A}$), which is assumed to be well below the lengthscale of interest here. At subsonic propagation (tip velocities below the LSRR) the stress field singularity coincides with the crack tip, while supersonic propagation separates the two. This is because the stress field at the tip cannot relax sufficiently fast and the singularity of the field then *lags behind*. This separation may persist even when the the tip velocity drops below the LSRR since it takes some time for the propagating stress field to catch up with the tip. The stress amplitude at the crack tip is therefore exactly that of the field at a distance that equals the seperation between the crack tip and the singularity. Thus we reinterpret Eq. (2.1) as the relation between the tip stress and the tip location relative to the moving field singularity. The stress at the tip, σ_{tip}, is obtained from (2.1) by putting $\zeta = (l - ct)\Theta(l - ct)$, where l is the tip's position along its path and $\Theta(l - ct)$, the Heavyside function, ensures that when the stress field front does overtake the tip ($l - ct \to 0^+$) then the stress converges to the static value in the moving frame and remains at that value for $l < ct$. The behavior of the crack propagation for $l < ct$ ($\Theta = 0$) is well understood [18] and need not be repeated here. We shall therefore focus on the case $\Theta = 1$ (i.e., when the tip precedes the singularity). For a propagation mode that mixes both $\Theta = 0$ and $\Theta = 1$ one simply pieces the analytic solutions together along the path. It should be made clear that this formulation does not require that the tip propagate at a straight line: The local tip velocity, $v = \dot{l}$, is taken *along* the crack path irrespective of the path's geometry. Differentiating Eq. (2.1) with respect to time, we obtain

$$\dot{\sigma}_{tip} = -\dot{\zeta}a^2/[\zeta(\zeta + 2a)]^{3/2} \ . \tag{2.2}$$

Henceforth we will omit the subscript from the tip stress and use σ instead. Expressing ζ in terms of σ from Eq. (2.1), rewriting $\dot{\zeta} = v - c$, and remembering that $v = v(\sigma)$, we can invert Eq. (2.2) to give

$$\dot{\sigma} = -[v(\sigma) - c](\sigma^2 - 1)^{3/2}/a \ . \tag{2.3}$$

This is the equation of motion for the propagation of the crack tip. This equation can be readily solved by quadratures to obtain the time dependence of σ:

$$t - t_0 = -\frac{a}{c}\int_{\sigma(t_0)}^{\sigma(t)} \frac{ds}{u(s)(s^2 - 1)^{3/2}} \ , \tag{2.4}$$

where $u(\sigma) = [v(\sigma) - c)/c]$ is a dimensionless quantity that we call the reduced velocity. Relation (2.4) describes exactly the kinetics of the tip stress and is the basis of our analysis. After finding $\sigma(t)$ from this expression, one can obtain $v(t) = v(\sigma(t))$ and then $l(t)$ and $\zeta(t)$. Note that the solution (2.4) is an exact derivation from the Yoffe solution and the only assumption that goes into the derivation is that the local propagation rate depends on σ alone. When v depends on other variables (e.g., explicitly on time) the equation of motion (2.3) still holds true but Eq. (2.4) needs to be modified.

The analysis allows for the following reinterpretation of the stress intensity factor, and its dependence on time, velocity, and history: In experiments one measures the stress at some small fixed distance δ ahead of the propagating crack tip (for clarity, we ignore here azimuthal dependence), which usually depends on the measurement method. Measured from the moving singularity, this distance is $\zeta = \delta + c\int u(t)dt$. By writing the dimensionless stress at that point both in terms of K and the Yoffe expression (Eqs. (1.1) and (2.1)) we get

$$\sigma(r) = (\zeta + a)/\sqrt{\zeta(\zeta + 2a)} \equiv K/\sigma_\infty\sqrt{2\pi r} \ . \tag{2.5}$$

A simple manipulation then leads to the time and velocity dependence of the stress intensity factor,

$$K = K_0\frac{1 + a/\zeta}{\sqrt{1 + 2a/\zeta}} \ , \tag{2.6}$$

where $K_0 = \sigma_\infty\sqrt{2\pi\delta}$ is a constant. Expressing the stress intensity factor explicitly in terms of $\zeta(t) = [l(t)-ct]\Theta(l(t)-ct)$, it becomes clear now why the propagation is observed to be time- and history-dependent. This suggests that relation (2.6) should be helpful in checking this theory against experimental measurements of K.

3. Analysis of noise-free propagation in the EC

Let us proceed now to analyse in detail the solution to the equation of motion (2.4). To make progress, we need information on the form of the local velocity response to the stress, $v(\sigma)$. First, crack propagation has been observed to be hysteretic in the following sense: σ_h, above which crack propagation starts (initiation), and $\sigma_l < \sigma_h$, to which the stress has to drop for the crack to halt its motion (arrest) [5][6]. The two thresholds, σ_l and σ_h, are material-dependent. Second, for stresses higher than the crack initiation value, the velocity appears to increas very slowly with stress [19]. Third, after initiation, considerable changes in K seem to have very little effect on the velocity [20]. All these observations can be combined to yield a generic form of $v(\sigma)$, which is plotted in Fig. 1. Along the upper branch of the hysteretic curve the velocity increases slowly with σ, while along the lower branch the velocity can be either identically zero (complete arrest) or very small (creep). As it turns out (see below), the dynamics of the propagation is determined by the relative positions of c and v_l on this generic response curve. It is therefore convenient to classify the behavior in terms of the dimensionless material-dependent parameter $\lambda \equiv c/v_l$. We note in passing that the slope of $dv/d\sigma$ at σ_l may or may not diverge. Both cases are discussed below.

As mentioned above, the medium in which the crack propagates is never really noise-free. What we mean by this term is that fluctuations in the local parameters and fields are negligible on time scales that are relevant to: i) the equation of motion, and ii) to measurement times. That is to say, there may well exist rapid fluctuations on short timescales due to phonons and other vibrational excitations, but are either unobservable by instruments that measure macroscopic crack propagation or they do not affect the slower mesoscale dynamics. This is reminiscent of the idea of fast versus slow degrees of freedom, where the fast ones are random and of small amplitude. This description, and Eq, (2.1), are valid when the number of microcracks is large enough to effect rapid fluctuations and the microcrack sizes are sufficiently small so that an individual fluctuation due to a large microcrack cannot be measured. Effectively, this amounts to considering the propagation process within an effective continuum (EC) whose properties are *different* than the bulk material, and where the motion of the tip is given by Eq. (2.1). The generalization to noisy dynamics, where large variations in the sizes of inhomogeneities lead to measurable fluctuations of the stress at the tip, will be discussed in detail in section 6.

In principle there are three cases that need to be considered in the noise-free context because they lead to different propagation behaviors:

Case I: $\boldsymbol{\lambda > 1}$

For materials that satisfy this relation, the value of c corresponds to a point, $F = (\sigma_c, c)$ which is located on the upper branch of $v(\sigma)$. Inspecting the integrand on the right-hand-side of Eq. (2.4), we see that it is regular at F. The behavior at the vicinity of this point can be found then by a linear analysis, which readily yields

$$|\sigma - \sigma_c| \approx Ce^{-\gamma\tau} \quad ; \quad \gamma \equiv \frac{(\sigma_c^2 - 1)^{3/2}}{a} \left(\frac{dv}{d\sigma}\right)_{\sigma_c} \quad . \tag{3.1}$$

In expression (3.1) we observe that the derivative of $v(\sigma)$ is continuous and positive in the neighborhood of F, which means that γ is positive. Thus, if the system is at a state near F, it will converge to this point at a rate determined by γ. It follows that F is a *stable fixed point* of the equation of motion for all λ.

In the units that we have chosen $\sigma_c > 1$ and $0 < (dv/d\sigma)_{\sigma_c} < 1$ [19][20]. To estimate the value of γ we recall that: i) the continuous description assumes a lengthscale above the cohesive zone, $r_0 \sim 5$nm, and ii) the typical velocity of the steady state in, say, PMMA is $c \sim 500$m/sec. Thus time is measured in units of $r_0/c \sim 10^{-11}$sec. It follows that the relaxation to the steady state from a state in the vicinity of F is so swift that small fluctuations are not only practically invisible by current measuring techniques, but they also occur at time scales that are only two or three orders of magnitude above the atomic vibrations. Although current data supports a low value of $dv/d\sigma$ along the upper branch, it is very unlikely that it can be so low as to change this conclusion and significantly reduce the value of γ. A typical relaxation of the system to the steady state from an initial stress higher than σ_c is shown in Fig. 2.

The stable solution that has just been described should be interpreted as a settling of the crack tip into a steady propagation rate, c, during which the local stress at the tip is constant at σ_c. We claim that this is the steady state that is frequently observed in experiments. We draw attention to the fact that the limiting speed of the steady state coincides with the local LSRR c. This has an important implication when it comes to interpretations of experimental data. In particular, it differs from the traditional view that this speed is the same as the speed of the Rayleigh waves along the fracture surfaces.

Another piece of information which can be obtained from the literature involves the location of F on the curve of $v(\sigma)$. It has been reported that immediately after crack initiation the stress intensity factor, K, drops and then stabilizes at a fixed value [5][6][19], a phenomenon that has been attributed to inertial effects [5]. However, in view of the above, this may simply suggest that in those materials the system slides down along the upper branch after crack initiation, until the fixed point F is reached and the stress saturates to

σ_c. In these systems then $\sigma_h > \sigma_c > \sigma_l$. Moreover, the fact that this decrease is observable on measurements timescales indicates that either or both scenarios occur: i. the interval $\sigma_h - \sigma_c$ is quite large, and ii. $dv/d\sigma$ is indeed small along the upper branch. We therefore suggest here that a detailed analysis of the time dependence of the drop in the tip stress after initiation is likely to yield the actual form of $v(\sigma)$ along the upper branch near and above the stable fixed point. Note, though, that in materials where $\sigma_c > \sigma_h$ the system will slide *upward* after crack initiation, which again can be utilized to chart the constitutive $v - \sigma$ relation.

A word of caution: the above interpretation is based on the noise-free analysis. It is shown below that noise may drive the system into a new steady state whose *measured* limiting velocity is lower than c. Since a cracking process is never noise-free (as evidenced by the fact that we usually hear fracture, see Section 6 for a detailed discussion about inherent noise in this system), the limiting velocity should be carefully interpreted, taking into consideration the history of $v(t)$.

Case II: $\boldsymbol{\lambda = 1}$

This is a marginal case that has only theoretical relevance when no fluctuations are present at all. The point F is at $F_l = (\sigma_c = \sigma_l, c = v_l)$ and the propagation dynamics depends strongly on the behavior of $(dv/d\sigma)_{F_l^+}$, where $F_l^+ = \lim_{\sigma\to\sigma_l^+}(\sigma, v)$. If this derivative is regular (finite) as F_l is approached from above the previous analysis applies unchanged and the approach to the fixed point follows relation (3.1). An approach from below (i.e., from smaller values of σ) is not possible because $\sigma < \sigma_l$ corresponds to a lower-branch state. Thus, coming from below, the system has to increase its stress along the lower branch, jump to the upper branch and then slide down to F. Thus we have a 'one-sided stability'. There is no apriori reason, however, why $(dv/d\sigma)_{F_l^+}$ should be finite. When it does diverge the dynamics of the crack tip are sensitive to the precise rate of that divergence. For illustration, consider the following form for $v(\sigma)$ near σ_l:

$$\begin{aligned} v(\sigma) &= v_l + v_0\left(\frac{\sigma}{\sigma_l} - 1\right)^\eta \\ u &\sim \frac{v_l - c}{v_0} + (\sigma - \sigma_l)^\eta \end{aligned} \tag{3.2}$$

where $0 < \eta < 1$ is a dimensionless number and v_0 is a prefactor with dimensions of velocity. This relation can be regarded as an approximation to a frequently occurring functional form in fractrure

$$v = v_l + v_0\left(1 - \exp\left[-\left(\frac{\sigma}{\sigma_l} - 1\right)^\eta\right]\right) \quad ; \quad \sigma \geq \sigma_l \ .$$

The behavior near the fixed point can be calculated from Eq. (2.4) and yields

$$\sigma - \sigma_c \sim (\tau_0 - t)^{1/(1-\eta)} \tag{3.3}$$

and

$$u \sim (\tau_0 - t)^{\eta/(1-\eta)} \ , \tag{3.4}$$

where $\tau_0 > t$ is some characteristic time whose value is determined by the properties of the material under consideration. This solution describes again a convergence of the velocity of propagation to $c = v_l$ but, unlike the former case, the convergence now is at a *power law* rate rather than exponentially. Therefore, this point can be regarded as only marginally stable. This marginal state is extremely sensitive to noise because when the crack propagation is close to the fixed point (σ_l, v_l) very small fluctuations can knock the system over the edge down to the lower branch.

Case III: $\lambda < 1$

Let us consider the case $c < v_l$ qualitatively first. Suppose that at a given moment the crack is propagating at speed v, which corresponds to a state along the upper branch. Since $v > c$ the tip is propagating faster than the relaxation rate of the stress field and the tip stress decreases steadily. This can be seen from relations (2.3) and (2.4) that show that the system will slide down the upper branch with the velocity and the stress at the tip dropping simultanuously. Unlike the previous cases, in the absence of a fixed point, the system is unable to converge to a steady state and, on reaching the point $F_l = (\sigma_l, v_l)$, it goes over the edge and drops to the lower branch. There the crack settles into a temporary arrest (if $v_{lb} = 0$) or a very slow creep (if $v_{lb} > 0$). Once on the lower branch, the stress front that have been lagging behind the tip start to catch up and the stress at the tip builds up. The increase in stress pushes the system up along the lower branch of the velocity-stress graph until it reaches the crack-initiation value, σ_h. At this moment the crack jumps back to the upper branch and the crack bursts away again. Propagating now at $v > c$, the crack tip is faster than the stress relaxation front and moves away from it, causing both the tip stress and the tip velocity to decrease again. The system slides then down the upper branch and the cycle repeats itself. This repetition gives rise to a periodic propagation behavior which in other contexts is usually termed a relaxation cycle.

With this picture in mind, we now wish to calculate quantitatively the characteristic times that are involved in the periodic process. A full cycle is completed at a period of time T that consists of the time that the system spends moving from σ_h to σ_l along the upper branch, and then in the opposite direction along the lower branch. Using relation (2.4) we can write T as

$$T = \frac{a}{c} \int_{\sigma_l}^{\sigma_h} \left[\frac{u_{ub}^{-1}}{(s^2-1)^{3/2}} + \frac{-u_{lb}^{-1}}{(s^2-1)^{3/2}} \right] ds \ , \tag{3.5}$$

where u_{ub} and u_{lb} are, respectively, the reduced velocities along the upper and lower branches. Note that u_{lb} is negative ($v < c$) and therefore both the terms in the integrand are positive. A complete crack arrest on the lower branch ($v_{lb} = 0$) corresponds to $u_{lb} = -1$, while creep corresponds to $u_{lb} = -1+\epsilon$, where $\epsilon > 0$. For illustration, consider a constitutive relation where

$$v_{lb} = 0 \quad ; \quad V_{ub} = v_0(\sigma^2 - A)/(\sigma^2 - 1) \ , \tag{3.6}$$

where $A = (v_0 + c/v_0$. The first term on the right hand side of (3.5) yields

$$T_{ub} = \frac{a}{2c} \ln \left(\frac{\alpha - y_l}{\alpha - y_h} \frac{\alpha + y_h}{\alpha + y_l} \right) \ , \tag{3.7}$$

where $\alpha = \sqrt{v_0/c}$, $y_k = -\sigma_k/\sqrt{\sigma_k^2 - 1}$ and $k = h, l$. The second term on the right hand side of (3.5) yields

$$\begin{aligned} T_{0,l} &= \frac{a}{c} \int_{\sigma_l}^{\sigma_h} \frac{ds}{(s^2-1)^{3/2}} = \\ &= \frac{a}{c} \left(\frac{\sigma_l}{\sqrt{\sigma_l^2 - 1}} - \frac{\sigma_h}{\sqrt{\sigma_h^2 - 1}} \right) \ . \end{aligned} \tag{3.8}$$

Thus the total period is

$$\begin{aligned} T &= T_{lb} + T_{ub} = \\ &= \frac{a}{c} \left[y_h - y_l + \ln \sqrt{\frac{\alpha - y_l}{\alpha - y_h} \frac{\alpha + y_h}{\alpha + y_l}} \right] \ . \end{aligned} \tag{3.9}$$

Typical time series of the velocity and tip stress in this periodic regime are shown in Fig. 3, alongside the steady-state behavior, for comparison.

For $\epsilon \ll 1$ (slow creep) we can expand the expression for the time spent on the lower branch and find that to first order in ϵ

$$T_l = T_{0,l}(1+\epsilon) \ .$$

When ϵ is of order 1 the lower branch does no longer describe slow creep. Rather, in this case the system may experience not only two velocities for a given value of the stress but also two stresses for a given velocity, which happens when $v(\sigma_h) > v(\sigma_l)$. The occurrence of this interesting situation may in fact be supported by observations in PMMA where low values of velocity fluctuations seemed to be well above zero [10]. This scenario can be studied in a straightforward manner along the lines outlined in the above analysis. We avoid going down this path in order to keep this presentation clear.

To summarize this section, the dynamics of the tip in the noise-free regime is characterized either by a steady-state propagation at the limiting velocity c, or by a periodically oscillating propagation rate with a period T, which has been found explicitly. We emphasize that there is only one material dependent parameter, λ, that controls which of these modes will come into play. The marginal propagation mode, although exisiting in principle according to the equation of motion, should be practically impossible to detect due to the inevitable noise in real systems.

4. Extension to Mode III propagation

Before we move to discuss effects of noise, we would like to point out that the above analysis can be readily generalized to Mode III propagation [21] and to a stress dependent LSRR. The results for mode III propagation remain unchanged qualitatively with the only difference manifesting in the explicit form of the solution for $\sigma(t)$. The expression for mode III propagation in two dimensions is

$$\sigma_{yz}|_{y=0} = K^* / \sqrt{2\pi[l - \int cdt]} \ , \tag{4.1}$$

where $\int cdt$ is the position of the stress singularity. Since the propagation is supersonic the tip is ahead of the stress singularity and the tip stress is

$$\sigma = \frac{K^*/\sqrt{2\pi}}{\sqrt{l(t) - \int cdt}} \ . \tag{4.2}$$

The term under the radical is the location of the tip in the frame of the moving stress field singularity, which again propagates at the local LSRR c. The value of c may, or may not, be constant, an issue that will be discussed in the next section. Differentiating Eq. (4.1) with respect to time we have

$$\dot{\sigma} = -\left(\frac{K^*}{2\sqrt{2\pi}}\right) \frac{(v-c)}{2\left[l(t) - \int cdt\right]^{3/2}} = -\frac{\pi\sigma^3(v-c)}{K^{*2}} \ . \tag{4.3}$$

As in mode I, this expression can be inverted to solve for $t(\sigma)$:

$$t - t_0 = -\frac{K^{*2}}{\pi} \int_{\sigma_0}^{\sigma} \frac{ds/s^3}{v(s) - c} \ . \tag{4.4}$$

For constant c, the above analysis for Mode I carries over to this case with essentially similar conclusions, namely:

1. For $\lambda > 1$, Eq. (4.4) has a fixed point on the upper branch at $v(\sigma_c) = c$. A linear stability analysis around the fixed point yields that it is stable with

$$\delta\sigma \sim e^{-\gamma t} \quad ; \quad \gamma = -\pi\sigma_c^3 \left(\partial v/\partial\sigma\right)_{\sigma_c} / K^{*2} \ .$$

Therefore, when the system is on the upper branch, it flows to the fixed point both from $v > c$ and $v < c$. As in mode I, the stable fixed point corresponds to the crack propagating at a steady state velocity c.

2. For $\lambda < 1$, the propagation consists of a series of relaxation cycles. The period of oscillation can be calculated explicitly again and turns out to be:

$$T = \frac{K^{*2}}{c\pi} \int_{\sigma_l}^{\sigma_h} \left(u_{ub}^{-1} - u_{lb}^{-1}\right) \frac{ds}{s^3} \ , \tag{4.5}$$

where u_{ub} and u_{lb} are the upper and lower branch reduced velocities, as before.

3. When the LSRR is velocity-dependent, $c = c(v)$, the analysis follows exactly the one carried out below for Mode I propagation.

5. Extension to non-constant LSRR

Suppose now that the local speed of sound is *not constant* but is rather a function of the local tip speed, $c = c(v)$. For simplicity we assume here that it is a function of v alone. A strong dependence on the propagation rate makes sense for the following reason: The higher the propagation rate the larger the damage that develops in front of the crack. The increased damage enhances both the scattering of sound waves and the nonlinear behavior in the neighborhood of the tip [8]. Thus, the LSRR is likely to depend on v. Both these effects (the damage in the PZ and the nonlinearity) act to reduce the local LSRR, which leads to $\partial c/\partial v \leq 0$ along the upper branch. Along the lower branch no damage is generated, implying that $c(v_{lb}) = c_0$ is constant. Since the tip velocity depends only on the tip stress we can then write $c(v(\sigma)) = c(\sigma)$. Eqs. (2.2) and (2.3) are not affected by the dependence of c on σ and the only difference is in the inversion of Eq. (2.3):

$$t - t_0 = -a \int_{\sigma(t_0)}^{\sigma(t)} \frac{ds}{[v(s) - c(s)](s^2 - 1)^{3/2}} \ . \tag{5.1}$$

Although this expression looks similar to (2.4), the nature of the solution needs to be carefully reconsidered. For a fixed point (denoted as σ^*, to distinguish it from the previous σ_c) to exist at all, the time-derivative of the stress (Eq. (2.3)) has to vanish at σ^*, indicating that $v^* \equiv v(\sigma^*)$ must be equal to $c^* \equiv c(\sigma^*)$. In other words the graph of $c(\sigma)$ has to intersect the upper branch of $v(\sigma)$, as sketched in Fig. 4 (full line). If such an intesection point exists, then a linear analysis near the fixed point gives

$$\delta\dot{\sigma} = -\frac{(\sigma^{*2} - 1)^{3/2}}{a} \left[v'(\sigma^*) - c'(\sigma^*)\right] \delta\sigma \ , \tag{5.2}$$

where the prime denotes derivative with respect to σ. It follows that

$$\begin{aligned} \delta\sigma &= \delta\sigma_0 e^{-\gamma t} \\ \gamma &= \frac{(\sigma^{*2} - 1)^{3/2}}{a} \left(\frac{dv}{d\sigma}\right)_{\sigma^*} \left(1 - \frac{\partial c}{\partial v}_{v^*}\right) \ . \end{aligned} \tag{5.3}$$

This expression is more involved than (3.1) in that it contains an extra term due to the variability of c. Since $\partial c/\partial v \leq 0$, as discussed, γ is positive, which determines the nature of the fixed point as stable, if it exists at all. As before, this fixed point represents a steady-state propagation at a speed c^*.

Now, suppose that $c(\sigma)$ *does not* intersect the upper branch of $v(\sigma)$, but rather lies underneath it (dashed line in Fig. 4), so that the two curves cross paths in the 'forbidden' zone between the branches. In this case the tip stress and velocity cannot saturate to a steady-state value because $\dot{\sigma}$ never vanishes, as can be seen by inspecting relation (2.3). Consequently, a periodic oscillatory behavior ensues, whose analysis is similar to the one carried out above for the constant-c Yoffe solution. The periodicity of the oscillation can be calculated using Eq. (5.1) between σ_l and σ_h.

A unique behavior that is worth mentioning is the following: even when there is an intersection point between $c(\sigma)$ and $v(\sigma)$ at some point along the upper branch of $v(\sigma)$, there can still occur a situation whereupon the tip does not settle into a steady-state propagation. This happens when the fixed point (c^*,σ^*) is *unstable*, namely, when $\gamma < 0$ in relation (5.3). Such a case may arise for materials where c *increases* with v at a sufficiently high rate, $\partial c/\partial v > 1$ in our units. In such materials small fluctuations will throw the system away from the fixed point and, depending on the direction of the fluctuation, the tip will either accelerate, ending up in a catastrophically running state, or decelerate until eventually dropping to the lower branch and halting altogether. As we shall see below, we expect stress fluctuation to act only to reduce the local tip stress (e.g., due to association of local microcracks along the path of the main crack) and therefore this scenario is most likely to result in an intermittent or quasi-periodic propagation, as analysed below. This scenario is not expected to occur in usual isotropic materials unless the value of c is governed by physical mechanisms other than those considered here (i.e., damage and nonlinearity).

6. Effects of noise on the dynamics in the EC mode

We now proceed to analyse effects of noise on the above results. First, let us consider the possible origins of fluctuations in this system. Even in the absence of external (e.g., thermal) fluctuations, the medium vibrates on short wavelengths. The reason is that the crack grows through a series of bond breaking events that give rise to violent vibrational excitations. These excitations are nonlinear due to two effects: 1. the nonlinearity of the interatomic potentials (recall that in front of the tip the atoms are highly strained and therefore a linear Hookian approximation is inapplicable), and 2. due to the disordered structure on the atomic scale in the cohesive zone. The nonlinearity manifests in localization and coupling between different modes. Moreover, whereas linear excitations (phonons) propagate fluently from the excited tip, the lattice nonlinearity leads to a nontrivial 'leaking' of energy from that zone. This, in turn, gives rise to an energy build-up in front of the propagating crack tip, as has indeed been observed in simulations [22]. However, having said all that, we argue that these effects are expected to dominate on lengthscales that are at most a few nm and therefore cannot persist to higher lengthscales mostly due to localization effects. If follows that this noise is irrelevant on the lengthscales discussed here. Thus, noise in the mesoscale comes mainly from the spatial and statistical size distribution of microcracks. For the purpose of this presentation, the term microcracks refers to crack-like inhomogeneities that occur in the PZ and which are much smaller than the main crack. The material inhomogeneities grow in response to the enhanced strain in front of the propagating crack and therefore their size and spatial distributions not only plays a significant role in the dynamic propagation but also the very nature of these distributions is determined self-consistently by the very same dynamics. Here we take for simplicity these distributions as given and proceed to consider the implications in the current theory. In principle, the fluctuations experienced by the crack tip upon encountering microcracks along its path may affect the consitutive relation $v(\sigma)$, as well as $\sigma(t)$. To keep the model simple, it is assumed in the following that $v(\sigma)$ remains unchanged on the time and length scales that are relevant to measurements, and only the local stress at the tip is affected. This simplifying assumption can be omitted in a more detailed calculation without running into a severe complication. However, such a modification requires a better understanding of the constitutive relation, which is yet to emerge.

The basic picture then is that the main crack propagates until it encounters a microcrack, at which moment the tip stress and the tip position change discontinuously. The positional change is due to the fact that the crack will resume propagation from *another point* along the boundary of the associated microcrack. Thus the location of the tip jumps discontinuously from one point along the microcrack boundary to another. The tip stress drops discontinuosly because the stress at the new point of propagation is lower than the stress at the original point. We can distinguish between two regimes: One, when the microcrack sizes are sufficiently small that the discontinuous positional changes are undetectable on measurable scales. In this regime only stress drop events need be considered while the small jumps in the tip's position are undetectable and therefore can be ignored. The case when microcrack are sufficiently large so that positional changes are detectable will be treated elsewhere. We only remark that in this case a detailed statistical analysis is necessary which is a straightforward extension of the analysis presented here. Nevertheless, we show that even without this modification one obtains a very rich behavior. In the following we will separate between the cases $\lambda > 1$ and $\lambda < 1$.

I. $\boldsymbol{\lambda > 1}$: *Fluctuations in the steady state*

The sensitivity of the dynamics in this case to fluctuations in σ depends on the noise amplitude and $\Delta_0 = \sigma_c - \sigma_l$, the difference between the fixed point and arrest stresses. A central quantity is the probability density of fluctuations in the tip stress amplitude, P_δ and it is therefore convenient to classify the behavior by the amplitude, A, and frequency of occurrence of fluctuations, ω.It is also convenient to define a system characteristic frequency, ω_0. This frequency corresponds to the inverse of the time that it takes the tip stress to increase from σ_l to σ_c,

$$\omega_0^{-1} = \frac{a}{c} \int_{\sigma_l}^{\sigma_c} \frac{dx}{u_{ub}(x^2-1)^{3/2}} \ .$$

Case I.1: $A < \Delta_0$; $\omega \ll \omega_0$

Consider first a very low occurrence frequency of fluctuations in σ and assume that there are no fluctuations whose amplitude is bigger than Δ_0, namely,

$$\mathrm{Prob}(A > \Delta_0) = \int_{\Delta_0}^{\infty} P_\delta(x)dx = 0 \ .$$

Physically, long time between successive fluctuations originates from a dilute spatial density of microcracks, while small amplitude of fluctuations stem fropm small sizes of the microcracks. A fluctuation pushes the system away from the fixed point to a new state of stress and velocity on the upper branch between σ_c and σ_l. From the new point the system moves back to the fixed point following kinetics that are governed by Eq. (2.4), whereupon it settles back into the steady state and awaits a new fluctuation. The probability density of the time intervals that it takes the system to settle back into the fixed point, τ, can be calculated directly from P_δ and relation (2.4):

$$\begin{aligned} P(\tau) &= P_\delta\left(\delta\sigma\right)\left|\delta\dot{\sigma}(\tau)\right| = \\ &= \frac{c}{a}P_\delta\left(\delta\sigma\right)\left|u_{ub}(\delta\sigma)[(\sigma_c - \delta\sigma)^2 - 1]^{3/2}\right| \ , \end{aligned} \tag{6.1}$$

where $\delta\sigma$ is a function of τ which is obtained from (2.4). Thus, this case displays small fluctuations around the steady state propagation rate. The probability density function of the fluctuations is given by (6.1). We can now provide a more precise criterion for what can be considered low occurrence frequency of the stress fluctuations: The time between successive fluctuations should be sufficiently long to let the system relax back to the fixed point. Therefore the distribution of the time intervals between two successive fluctuations, θ_n, ahould satisfy $P(\tau > \theta_n) \ll 1$.

Case I.2: $A < \Delta_0$; $\omega \gg \omega_0$

Consider now the other extreme when the occurrence frequency of the tip stress fluctuations is very high, $\omega \gg \omega_0$, but keeping the amplitude, A, below Δ_0. This corresponds to small microcracks, that populate densely the PZ. In this regime, once the system has been kicked away from the fixed point, the tip is typically not allowed sufficient time to recover before a new fluctuation appears that further reduces the stress. Several fluctuations in quick succession then push the system all the way down to σ_l and over the edge to the lower branch. Given sufficiently fast measurements, the crack will be observed to propagate haltingly until it stops when the system drops to the lower branch. Once there, the stress builds up, as described already, until a new initiation occurs. The resulting behavior of the system can be either chaotic, intermittent or even seemingly periodic, depending directly on the statistics of the noise. To illustrate the statistical richness, consider the following kinetics: The system has just jumped to the upper branch at σ_h and starts sliding down according to the EC equation of motion (following equations (2.4), (4.4), or (5.1)). After a period of time θ_1, during which the tip stress decreased continuously to $\sigma(\theta_1)$, a fluctuation of size $\delta\sigma_1$ occurs. The stress drops discontinuously to $\sigma(\theta_1) - \delta\sigma_1$ and the system resumes continuous sliding according to the EC equation of motion until another fluctuation comes along. We would like to analyse the statistics of the propagation dynamics that result from this series of events. In particular, we wish to study the statistics of the time, T_u, that the system spends on the upper branch before it reaches σ_l. If we regard the process as a time series of fluctuations of sizes A_n, each following a quiet period of θ_n, the total time that the system spends on the upper branch is

$$T_{ub} = \frac{a}{c} \sum_{n=1}^{N} \int_{\sigma_n}^{\sigma_{n-1}-A_{n-1}} \frac{dx}{u_{ub}(x^2-1)^{3/2}} , \tag{6.2}$$

where $\sigma_{n=0} = \sigma_h$, $A_{n=0} = 0$, σ_n is the stress that the system reached starting from $\sigma_{n-1} - A_{n-1}$ following the EC dynamics for a period of time θ_n, and N is the total number of fluctuations needed to drop all the way down to σ_l. In expression (6.2), note the difference in the kinetics above and below σ_c: For $\sigma > \sigma_c$ the equation of motion reduces the stress between successive fluctuations, thus helping the downslide of the system. But once the stress drops below c the EC kinetics act to *increase* the local tip stress, opposing the downslide. This difference is important in that it gives rise to a continuous spectrum of noise-driven steady states which can occur only for $\sigma < \sigma_c$, as will be shown below. Back to the current calculation, after the system drops to the lower branch, the tip stress increases without interruption from σ_l to σ_h, a process that takes a period of time

$$T_{lb} = -\frac{a}{c} \int_{\sigma_l}^{\sigma_h} \frac{dx}{u_{lb}(x^2-1)^{3/2}} . \tag{6.3}$$

The period of an entire cycle is $T = T_{lb} + T_{ub}$ and the distribution of this quantity can be computed from the above expressions. It is interesting to note that the series for T_{ub}, (6.2), can be regarded as a variation on the known random walk process. A step in the latter is analogous here to a drop of A_n while the steady stress decrease in between stress jumps is analogous to a position-dependent drift between steps. A detailed statistical analysis will be carried out elsewhere. However, even without such an analysis, it can be surmised from relation (6.2) that the behavior would be sensitive to the distributions of θ_n and A_n. In particular, it would be interesting to assume the standard Weibull statistics for the distribution of the microcracks and study the statistics of the propagation dynamics.

Case I.3: $A < \Delta_0$; $\omega \approx O(\omega_0)$

So, small fluctuations at low occurrence frequencies hardly affect the steady state much, while high frequencies can lead to complex dynamic behaviors, depending on the distributions of θ_n and A_n. Let us turn attention now to intermediate values of θ, where we will find another surprise: New *dynamic steady-states.* Consider a system whose material parameter is $\lambda = \lambda_0 > 1$. Suppose that at time t_0 the system is at some point σ_0 on the upper branch such that $\sigma_l < \sigma_0 < \sigma_c$. Now let a fluctuation of size A_0 jolt the system down to $\sigma_0 - A_0 > \sigma_l$. According to the equation of motion, the stress will start increasing until, after a time θ_1 when it reaches a new stress σ_1, a new fluctuation, A_1, arrives which pushes the stress down again to $\sigma_1 - A_1$. The stress then increases again until the next fluctuation, and so on. Suppose now that the values of the time intervals between successive fluctuations, θ_n, are typically similar to the time that it takes for the tip stress to build up to about the same value that it started from. The tip stress can only build up then to approximately the initial stress before another fluctuation comes along and knocks it down again. Thus the stress will fluctuate repeatedly along the upper branch *below* σ_c, never quite reaching the fixed point yet never dropping to the lower branch. The average limiting speed will be then below the real LSRR in the PZ. A typical such propagation process is shown in Fig. 5, where the noise-free propagation is at $c_0 = 1.55$, while the noise-driven new state is at a distinctly lower mean velocity of $c_{\rm eff} = 1.52 \pm 0.02$.

Identifying such a steady state in real experiments is very significant to the interpretation of the propagation speed data. An explicit statistical analysis of the propagation history in this regime is currently under way.

Case I.4: $A > \Delta_0$; $\omega \ll \omega_0$

We have discussed the low A regime of the $A - \omega$ state diagram. We now turn attention to large fluctuations. In probabilistic terms this corresponds to

$$\mathrm{Prob}(A > \Delta_0) = \int_{\Delta_0}^{\infty} P_\delta(x) dx = O(1) \ .$$

Although the noise may be dominated by large amplitude fluctuations, small amplitude noise (background noise) may still exist, which will give rise to the effects discussed above. Thus the analysis here should be understood as superposing on the above behavior. Start the system at the fixed point. At low frequencies of the large amplitude, a large fluctuation, $A > \Delta_0$, comes along and reduces the stress to *below* σ_l and the system falls to the lower branch. There the propagation halts and the stress increases along the lower branch. As before, on reaching σ_h the system jumps to the upper branch and the tip resumes propagation. The time, T_1, that it takes for the tip to re-settle into the steady state is given by

$$\begin{aligned} T_1(A) = & \frac{a}{c} \int_{\sigma_c - A}^{\sigma_h} \frac{dx}{-u_{lb}(x^2-1)^{3/2}} + \\ & + \frac{a}{c} \int_{\sigma_c}^{\sigma_h} \frac{dx}{u_{ub}(x^2-1)^{3/2}} \ , \end{aligned} \tag{6.4}$$

where $\sigma_c - A < \sigma_l$ and both the integrals are positive. Using (6.4), the term 'low' for the occurrence frequency can be quantified as follows: The time between two successive large fluctuations, θ_n, has to be sufficiently long to let the system return to the steady state. This means

$$\int_0^{T_1} P(\theta_n) \ll 1 \ , \tag{6.5}$$

where $P(\theta_n)$ is the probability density of θ_n. Expression (6.5) states that there are hardly any time intervals θ_n that are smaller than T_1. The velocity history in this regime is *intermittent*, with occasional crack arrests between long periods of steady state propagation, as sketched in Fig. 6. It is not difficult to calculate the probability density of T_1 given the probability density of A, $P_\delta(A)$:

$$P(T_1) = \frac{c}{a} P_\delta(A) \left[(\sigma_c - A)^2 - 1\right]^{3/2} |u_{lb}(\sigma_c - A)| \tag{6.6}$$

where $A > \Delta_0$ is given explicitly in terms of T_1 by inverting relation (6.4).

For example, assuming complete arrest on the lower branch ($v_{lb} = 0$) we find from relation (6.4) that

$$A(T_1) = \sigma_c - \left[1 - \left(\frac{a/c}{T_1 - \tau_0}\right)^2\right]^{-1/2} , \tag{6.7}$$

where

$$\tau_0 = \frac{a}{c}\left[\int_{\sigma_l}^{\sigma_h} \frac{ds}{u_{ub}(s^2-1)^{3/2}} - \frac{\sigma_h}{\sqrt{\sigma_h^2 - 1}}\right] \tag{6.8}$$

is a constant. Substituting in relation (6.6), the probability density of T_1 is found to be

$$P(T_1) = \frac{c}{a}\left[\left(\frac{T_1 - \tau_0}{a/c}\right)^2 - 1\right]^{-3/2} P_\delta\left[A(T_1)\right] \ . \tag{6.9}$$

This probability density can be inferred from experimental measurements of the velocity history. This, in turn, can be used to both test this theory and learn about the distribution of the microcracks in the PZ.

Case I.5: $A > \Delta_0$; $\omega \gg \omega_0$

Next, consider large amplitude fluctuations at *high* occurrence frequencies. Following the same rationale as above, let us follow a typical trajectory: Start the system on the upper branch and let a large fluctuation knock it off it to the lower branch. The tip stress increases along the lower branch, which takes a period of time longer than T_{lb}. The excess time above T_{lb} can be determined from the fluctuation amplitude A. As the tip stress reaches σ_h the system jumps to the upper branch, as above. Now, however, the tip stress is not allowed to decrease continuously to the fixed point. Rather, a new large fluctuation quickly comes along and reduces the tip stress to a point on the lower branch that is considerably lower than σ_h. Since these fluctuations are very frequent, within a short time the system is forced to drop again to the lower branch, there to resume building up the tip stress and repeat the process. The period of time is $T = T_{lb} + T_{ub}$, where T_{ub} is now much smaller than T_{lb}. The fluctuating part is the time spent on the upper branch plus the time spent on the lower branch from immediately after the drop (at a stress somewhat lower than σ_l) until the stress builds up to σ_l. The time spent on the lower branch during the stress build-up from σ_l to σ_h is constant. If the probability density function of the amplitudes is not anomalously broad then the behavior in this regime is *quasi-periodic*, with the period time fluctuating around a value somewhat above T_{lb}. Such a behavior is shown in Fig. 7. It should be emphasized that this quasi-periodic propagation is driven *only* by the noise and without it the propagation would be at a steady state. We point out that if the amplitude of the fluctuations is widely distributed so will the period time and the quasi-periodic appearance may be washed out due to a large variation in the time spent along the lower branch before the stress reaches σ_l.

Thus the separation in the amplitude-frequency phase space of the fluctuations is not absolutely sharp. Large fluctuations with intermediate occurrence frequencies can lead to behaviors that range from completely chaotic, through intermittent to quasi-periodic, all depending on the interplay between the distributions of $\delta\sigma$ and θ. Nevertheless, for sharp distributions of these quantities, one can draw a propagation state diagram from the above analyses for $\lambda > 1$. This diagram has the form shown in Fig. 8.

The case $\lambda = 1 + \epsilon$, where $\epsilon \ll 1$, can be treated as a limiting case of the foregoing when $\Delta_0 \to 0$. In this case small fluctuations can knock the system to the lower branch and initiate a relaxation cycle. As discussed already, since fluctuations always occur, it is clear that no proper steady state propagation can exist for any value of Δ_0 that is below the background noise level. Therefore, for narrow distributions of the noise frequency and amplitude (namely, distributions whose tails decay exponentially) the system is expected to be mostly in a periodic-like propagation regime, with the statistics of the period time being computable from the distribution of the fluctuations, as outlined above.

II. $\boldsymbol{\lambda < 1}$: *Fluctuations in the periodic propagation*

Analysis of noise in the periodic mode is less complicated because, in the absence of a fixed point, the equation of motion always acts to *reduce* the stress on the upper branch. This is in contrast to $\lambda > 1$, where one has to take into account the different signs of $u(\sigma)$ above and below (σ_c, c). Here it is convenient to define a characteristic amplitude, $\Delta_1 = \sigma_h - \sigma_l$, and a characteristic frequency,

$$\omega_1^{-1} = \frac{a}{c} \int_{\sigma_h}^{\sigma_l} \frac{dx}{u_{ub}(x^2 - 1)^{3/2}} \ .$$

ω_1 is nothing but the inverse of the time it takes to slide along the entire upper branch in the noise-free periodic propagation mode.

Case II.1: $A \ll \Delta_1$

At low A and ω the system slides down the upper branch towards σ_l, performing a few small jumps that are separated by long periods of continuous decrease of the stress. Noise reduces somewhat the time that the system spends on the upper branch, but the time that the crack stays arrested along the lower branch remains unchanged compared to the noise-free case. Thus, the underlying periodicity still shows but the mean period of the relaxation cycle is slightly shorter. The distribution of the reduced period, $\delta T = T_{lb} + T_u$, can be readily analysed using expression (6.2).

Increasing ω while keeping A small reduces the time spent along the upper branch and increases the measured mean frequency of the periodic behavior. The effect of noise on the velocity for $\lambda \approx 0.85$ and two different noise frequencies is shown in fig. 9 alongside the noise-free history. The noise frequencies correspond to $\theta = 0.04$ and 0.2. Note the reduction in the period as the noise frequency increases. For $\theta = 0.2$ the period time decreases by 6% compared to the noise-free propagation, while for $\theta = 0.04$ the reduction is by 27%.

The statistics involved can be analysed along similar lines: Given the probability density of A_n and θ_n, one uses Eqs. (2.4) and (6.2) to calculate the probability density of the time spent along the upper branch within one cycle. The narrower those probability densities the sharper the distribution of the cycle periodicities and vice versa. For very low noise frequencies ($\ll \omega_1$) the behavior is almost perfectly periodic with occasional cycles of short periods. In other words, in this case it is the time series of the *frequency* of the velocity oscillation period which is intermittent rather than the velocity itself!

Case II.2: $A > \Delta_1$; $\omega < \omega_1$

Cranking up the noise amplitude in this regime, has two competing effects: On the one hand, this increases the jumps in the stress of the system along the upper branch and therefore reduces the time spent on it. On the other hand, it increases the probability that the system will drop to the lower branch at a stress value much lower than σ_l. The latter increases the time, T_{lb}, that the crack stays arrested after a fluctuation. Which effect is more dominant depends on the particular form of $v(\sigma)$. The net time delay due to arrest can be either longer or shorter than the time gain along the upper branch. Therefore, the overall relaxation cycle period can either increase or decrease. For very low occurrence frequencies of the large fluctuations (i.e., $\theta_n > T$) the process will consist of the usual periodic behavior, as in the noise-free case, with occasional long arrest periods. The arrest periods will become longer as the amplitude of the noise increases. The broader the distribution of the amplitudes of A_n, the broader Will be the distribution of the occasional long arrest periods. A generic propagation state diagram for $\lambda < 1$ is shown in Fig. 10. This diagram is not as rich as the corresponding one for $\lambda > 1$, but it should be born in mind that these diagrams presume very sharp distributions of A and ω, while real systems usually entertain distributions that are never sharp. A range of amplitudes and frequencies will lead to propagation histories that may consist of mixed states, complicating the identification of the propagation state from the history of the propagation speed.

Case II.3: $A > \Delta_1$; $\omega > \omega_1$

Turning up ω for the large fluctuations, the long arrest periods become more frequent and the behavior changes from periodic to intermittent. In contrast to the previous case, now the intermittency consists of occasional bursts of propagation between long periods of pauses. Since there are experiments that can yield accurate measurements of tip velocity oscillations and arrest lines [10], it is suggested here to use such measurements to infer, following the above analysis, the characteristics of the noise. By finding $P_\delta(A)$ and $P(\theta)$, and feeding this information back into the foregoing calculation, one can improve the accuracy of the statistical analysis presented here. Moreover, by applying the analysis to the measured data it should be Possible to deduce the statistical properties of the microcrack distribution in the processing zone ahead of the crack tip.

7. Summary, conclusions, and future directions

To summarise, in this paper we generalized and extended the theoretical model proposed in ref. [1] for the dynamics of fracture propagation in two dimensional amorphous and polycrystalline materials. This model obviates the use of the stress intensity factor altogether and explains the history-dependence of the propagation behavior without inventing new adjustable parameters. The theory is a direct consequence of the observed slow stress relaxation near the crack front and the different initiation and arrest stresses. The formulation presented here is flexible in that it allows the use of any form of the constitutive relation, $v(\sigma)$, between the local velocity and the local tip stress. Even without assuming any particular form of $v(\sigma)$ it was shown that it is possible to derive explicit results for the propagation dynamics that turn out to display a rich behavior. Generally, the behavior was found to depend only on the ratio of the stress relaxation rate, c, and the lowest speed before arrest. In the absence of noise, the propagation speed was shown to either saturate into a steady state or to oscillate periodically through a relaxation cycle. The effects of noise were studied in detail and were found to be quite diverse: Noise can either simply smear the aforementioned behaviors or it can give rise to completely new modes of propagation, such as intermittent oscillations, noise-driven periodic oscillations, or noise-driven new steady states. We argued that noise originates mainly due to the inhomogeneities in the processing zone ahead of the fracture and went on to analyse the effects of such noise on the previously derived noise-freer behavior. However, when microcracks in the processing zone are so large that Microcrack association becomes a dominant mechanism for propagation, then the spatial distribution of the microcracks and their sizes should be taken explicitly into consideration. We have outlined here how to analyse these statistics and the effects of large microvoids on the dynamics, paving the way for future such calculations. It useful to give a criterion for distinguishing between the two regimes: We note that dynamics through association of microcracks is significant when microcrack sizes are sufficiently large to affect observable velocity changes over $\sim 10^{-6}$sec. If the mean velocity of propagation is of order $\sim 10^{2}$m/s then only voids of hundreds of microns can give rise to this mode. Thus, in cases where microcracks do not reach these sizes we expect the above theory to hold true. Nevertheless, the theory can be readily extended to cases with very large microcracks in the processing zone by using the statistical treatment outlined here. The way to go about this is by considering the propagation as composed of two alternating processes: The first, propagation in a region of small microcracks, where the effective continuum applies. The second is propagation via an association event, whence a large microcrack is encountered and incorporated into the main crack. By considering the statistics of the microcrack sizes and the spaces between them, it is possible to derive the discontinuous drops in the tip stress as well as the jumps in the tip location immediately after association. The combination of the distances covered by continuous motion and jumps can then be analysed with the above formalism. Knowledge of the respective time series, given in expression (6.2), and a simultaneous comparison with the distances distribution yields the velocity history. This calculation is currently being carried out for various microcrack distributions and will be reported shortly.

There are still many aspects and implications of this model that remain to be explored. For example, the effects of particular noise statistics, the existence of statistical correlations in the noise, the explicit effects of particular distributions of microcrack sizes, and the implications of the velocity and void distributions on the roughness of the fracture surface that is left behind the tip. It is this author's belief that the latter issue is of extreme significance in that it can bridge between the dynamics and the quest for finding relations between the fracture roughness and material properties, such as toughness. In the absence of an adequate dynamical theory, these two aspects, which are clearly related, drifted into two seemingly independent fields of research. There is a strong possibility that the approach taken here can bring about such a connection and unify the inter-relations between material properties, dynamics, and roughness. All these issues are left for later studies.

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FIGURE CAPTIONS

1. A generic plot of $v(\sigma)$.
2. Stress relaxation towards the steady state.
3. A typical velocity history in the periodic regime for $\lambda = 0.8$. A steady state with $\lambda = 1.1$ is plotted for comparison.
4. A generic plot of $v(\sigma)$ with a non-constant LSRR: a. $v(\sigma)$ and $c(\sigma)$ intersect along the upper branch; b. $v(\sigma)$ and $c(\sigma)$ intersect in the forbidden regime.
5. A noise-induced new steady state. Without noise the steady state speed is $c_0 = 1.55$. With noise the mean propagation speed is $c_{eff} = 1.52 \pm 0.02 < c_0$.
6. An intermittent propagation with few arrests.
7. Quasi-periodic propagation.
8. A generic $A - \omega$ propagation state diagram for $\lambda > 1$. The propagation states are steady state (SS), quasi-periodic (QP), intermittent (I), and dynamic steady state (DSS).
9. Noise propagation in the periodic regime ($\lambda = 0.85$). Fluctuation are introduced every $t = 0.04$ (dotted line) and $t = 0.2$ (dashed line). Note the respective reduction of 27% and 6% in the period time compared with the noise-free propagation (full line).
10. A generic $A - \omega$ propagation state diagram for $\lambda < 1$. The propagation states are periodic (P) and intermittent (I).

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